LOCAL PROPERTIES OF GRAPHS

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Summary

We say a graph is *locally* $\mathcal{P}$ if the induced graph on the neighbourhood of every vertex has the property $\mathcal{P}$. Specifically, a graph is *locally traceable* (LT) or *locally hamiltonian* (LH) if the induced graph on the neighbourhood of every vertex is traceable or hamiltonian, respectively. A *locally locally hamiltonian* (L^2H) graph is a graph in which the graph induced by the neighbourhood of each vertex is an LH graph. This concept is generalized to an arbitrary degree of nesting, to make it possible to work with L^kH graphs. This thesis focuses on the global cycle properties of LT, LH and L^kH graphs. Methods are developed to construct and combine such graphs to create others with desired properties.

It is shown that with the exception of three graphs, LT graphs with maximum degree no greater than 5 are fully cycle extendable (and hence hamiltonian), but the Hamilton cycle problem for LT graphs with maximum degree 6 is NP-complete. Furthermore, the smallest nontraceable LT graph has order 10, and the smallest value of the maximum degree for which LT graphs can be nontraceable is 6.

It is also shown that LH graphs with maximum degree 6 are fully cycle extendable, and that there exist nonhamiltonian LH graphs with maximum degree 9 or less for all orders greater than 10. The Hamilton cycle problem is shown to be NP-complete for LH graphs with maximum degree 9. The construction of r-regular nonhamiltonian graphs is demonstrated, and it is shown that the number of vertices in a longest path in an LH graph can contain a vanishing fraction of the vertices of the graph.

Various properties of L^kH graphs are investigated, and it shown that nonhamiltonian L^kH graphs exist of order $9 + 2k$ for $k \geq 1$. The Hamilton cycle problem is shown to be NP-complete for L^2H graphs with maximum degree 12, and NP-complete for graphs that are both LH and L^2H with maximum degree 13. The

NP-completeness of the Hamilton cycle problem for $L^k H$ graphs for higher values of k is also investigated.

Key terms:

Graph theory; Hamilton cycle; Hamilton path; locally hamiltonian; locally traceable; vertex degree; nonhamiltonian; nontraceable; graph order; NP-complete

Declaration

I declare that Local Properties of Graphs is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references. I further declare that I have not previously submitted this work, or part of it, for examination at Unisa for another qualification or at any other higher education institution.

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Chapter 1

Introduction

1.1 Definitions

Except where otherwise indicated, the definitions to follow can be found in Bondy and Murty [9].

We limit ourselves to *simple graphs*, that is, graphs with at most one edge between any two vertices, no loops, and no directed edges. The set of edges of a graph G is denoted by $E(G)$ and the set of vertices by $V(G)$. For any set S, $|S|$ is the cardinality of S. We call $|V(G)|$ the *order* of a graph, and we often use $n(G)$ interchangeably with $|V(G)|$. We call $|E(G)|$ the *size* of the graph. We can refer to an edge between two vertices u and v as uv, and also use the notation $u \sim v$ to indicate that u and v are neighbours, while $u \not\sim v$ indicates that u and v are not neighbours. We use $N(v)$ to represent the open neighbourhood of a vertex v, and $N[v]$ for the closed neighbourhood. If there is room for ambiguity regarding to which graph we're referring, we use a subscript, for example, $N_G(v)$.

A *subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* on a set X of vertices in $V(G)$ is the graph obtained by starting with X and adding an edge between two vertices u and v in X if there is an edge between u and v in G. This is written as $\langle X \rangle$. The graph $G - X$ is the graph obtained by removing the vertices in X from G and all the edges incident to vertices in X.

The degree of a vertex v is the number of edges incident to v, and is denoted by $d(v)$. The maximum and minimum degrees of the vertices of G are denoted by

$\Delta(G)$ and $\delta(G)$, respectively, and if the graph we're referring to is clear from the context, we may just use Δ and δ. We will refer to $\Delta(G)$ and $\delta(G)$ as the *maximum degree* and the *minimum degree* of G.

Two graphs G and H are *isomorphic* if there is a bijection $\phi : V(G) \to V(H)$ such that two vertices in G are adjacent if and only if they are also adjacent in H.

A *complete graph* K_n is a graph on n vertices with an edge between any two vertices in $V(K_n)$. A k*-clique* in a graph G is a subgraph of G that is isomorphic to the complete graph K_k. An r*-regular* graph is a graph in which all vertices have degree r, where r is a nonnegative integer. A *planar graph* is a graph that can be represented in two dimensions in such a way that no edges cross. A k*-partite* graph is a graph whose vertex set can be partitioned into k subsets $V_1, V_2, \ldots, V_k$ such that no two vertices in any given subset are adjacent. A k*-partite* graph is *complete* if any two vertices that are not in the same subset are adjacent, and is denoted by $K_{n_1,n_2,\ldots,n_k}$, where n_i is the cardinality of subset V_i, $i = 1, 2, \ldots, k$.

A graph is *connected* if, for every partition of its vertex set into two nonempty sets X and Y, there is an edge with one vertex in X and one vertex in Y. A *path* is a simple graph whose vertices can be arranged in sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and not adjacent otherwise. We will use P_n to denote a path containing n vertices. Let $P = p_1 \ldots p_i$ and $Q = q_1 \ldots q_j$ be two paths in G. Then the concatenation of the two paths $p_1 \ldots p_i q_1 \ldots q_j$ is denoted by PQ. The *detour order* of a graph is the order of a longest path in the graph. A *cycle* is a graph of order at least 3 whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if and only if they are consecutive in the sequence. We will use C_n to denote a cycle of length n. The *girth* of a graph G is the length of a shortest cycle in G, and the *circumference* of G is the length of a longest cycle in G. A *component* is a subgraph in which any two vertices are connected by a path, and no vertex in the component is connected to a vertex outside the component. The number of components of a graph G will be denoted $comp(G)$. If X is a subset of $V(G)$, where G is a connected graph, such that $G - X$ is not a connected graph, then X is referred to as a *vertex cutset* of G.

Two paths that have the same end vertices but have no other vertices in common are called *internally disjoint*. A graph G is k-*connected* if, for any $u, v \in V(G)$ there are at least k internally disjoint paths with end vertices u and v. The *connectivity* κ of G is the maximum value of k for which G is k-connected.

A graph is *hamiltonian* if the circumference of the graph is equal to the order of the graph. A graph is *traceable* if the detour order of the graph is equal to the order of the graph. A cycle C in a graph G is *extendable* if there exists a cycle C' that contains all the vertices of C as well as one additional vertex of G. A graph G is *cycle extendable* if every nonhamiltonian cycle is extendable, and is *fully cycle extendable* if in addition every vertex lies in a cycle of length 3. A graph G is *chordal* if every cycle of length greater than three has a chord.

We say a graph G is *locally* $\mathcal{P}$ if $\langle N(v)\rangle$ has the property $\mathcal{P}$ for every vertex $v \in V(G)$. In particular, a graph is *locally connected* (abbreviated LC), *locally traceable* (abbreviated LT), and *locally hamiltonian* (abbreviated LH) if $\langle N(v)\rangle$ is connected, traceable, and hamiltonian, respectively.

If t is a positive real number, a graph G is t-tough if $comp(G - S) \leq |S|/t$ for every vertex cutset S of $V(G)$. The *toughness* of a graph G, denoted $t(G)$, is defined as $t(G) = \min\{\frac{|S|}{comp(G-S)}\}$, where the minimum is taken over all vertex cutsets S of G.

A set $U \subseteq V(G)$ is independent if there are no edges between vertices in U. The *independence number* of G, denoted $\alpha(G)$, is the cardinality of the largest independent subset of vertices in $V(G)$.

A connected graph that contains no cycles is called a *tree*. A generalized version of this concept is that of a k-*tree*. A k-tree is a graph that can be constructed in the following way: start with a complete graph K_{k+1}. The graph can be expanded by adding one vertex v of degree k at a time, with the requirement that the $\langle N(v)\rangle$ is a k-clique [28]. If a k-tree G is constructed in such way that no more than one vertex is added to any clique, then G is called a simple-clique k-tree (SC k-tree) [22].

For any graph H, a graph G is said to be H-*free* if G does not contain H as an induced subgraph.

The class of problems that are solvable in polynomial time is denoted by $\mathcal{P}$ [10]. A related class of problems is denoted by $\mathcal{NP}$, which stands for nondeterministic polynomial time. A problem is in $\mathcal{NP}$ if it is possible to confirm in polynomial time that a proposed solution is a valid solution, implying that $\mathcal{P} \subseteq \mathcal{NP}$. A problem is *NP-complete* if a polynomial-time algorithm for solving it would result in polynomial-time solutions for all problems in $\mathcal{NP}$.

The *Hamilton Cycle Problem* (which will be abbreviated to HCP when convenient), is the problem of deciding whether a graph is hamiltonian or not. We use the notation Δ^*_X to denote the maximum value of Δ for which the HCP for the class X of graphs can be calculated in polynomial time.

1.2 Background

This thesis focuses on two local properties, namely local traceability and local hamiltonicity, and how they relate to traceability and hamiltonicity. However, I think it is a good idea to start with an overview of local connectedness, to give the reader an insight into how the increasing strength of the local condition affects the properties of the graph. The concept of local connectedness was introduced by Chartrand and Pippert [11] in 1974, where they proved the following theorem.

Theorem 1.2.1. [11] *If G is a connected, LC graph of order at least 3 and $\Delta(G) \leq 4$, then G is either hamiltonian or isomorphic to the complete 3-partite graph $K_{1,1,3}$.*

Kikust [23] investigated the case where G is 5-regular.

Theorem 1.2.2. [23] *A connected, LC graph that is 5-regular is hamiltonian.*

Hendry [20] strengthened Kikust's theorem.

Theorem 1.2.3. [20] *Let G be a connected, LC graph such that $\Delta(G) \leq 5$ and $\Delta(G) - \delta(G) \leq 1$. Then G is fully cycle extendable.*

Gordon et al. [19] extended the range of vertex degrees of G for which G is fully cycle extendable.

Theorem 1.2.4. [19] *Let G be a connected, LC graph with $\Delta(G) = 5$ and $\delta(G) \geq 3$. Then G is fully cycle extendable.*

They also proved a useful theorem for when $\delta = 2$:

Theorem 1.2.5. [19] *If G is a nonhamiltonian connected, locally connected graph with $\delta(G) = 2$ and $\Delta(G) = 5$, then at least one of the following holds.*

(*a*) *$G \in \{M_3, M_4, M_5\}$ (see Figure 2.4).*

(*b*) *G contains two nonadjacent vertices x_1, x_2 of degree 2 such that $N(x_1) = N(x_2)$.*

(*c*) *G contains the graph F depicted in Figure 1.1 as induced subgraph.*

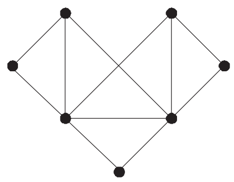

Figure 1.1: The graph F.

On the other hand, Gordon et al. [19] also showed that the Hamilton Cycle Problem is NP-complete for LC graphs with maximum degree 7. They thus showed that $4 \leq \Delta^*_{LC} \leq 6$, and speculated that the correct value is 6. However, at a workshop held at Salt Rock in January 2016 at which Susan van Aardt, Alewyn Burger, Marietjie Frick, Carsten Thomassen and I participated, we proved the following.

Theorem 1.2.6. [1] *The Hamilton Cycle Problem for LC graphs with $\Delta = 5$ and $\delta = 2$ is NP-complete.*

It follows that $\Delta^*_{LC} = 4$. I shall investigate the values of Δ^*_{LT} and Δ^*_{LH} in Chapters 2 and 3.

A graph is considered to be *claw-free* if the graph contains no induced $K_{1,3}$. This can also be seen as a local condition: a graph G is claw-free if $\alpha(\langle N(v) \rangle) < 3$ for all $v \in V(G)$. Combining this with local connectedness leads to a powerful result by Oberly and Sumner [25].

Theorem 1.2.7. [25] *Let G be a $K_{1,3}$-free, connected, LC graph of order at least 3. Then G is hamiltonian.*

Clark [13] showed that the conditions in Oberly and Sumner's theorem are sufficient to ensure that the graph G is pancyclic, and Hendry [21] noted that Clark had actually proved that G is fully cycle extendable.

In [25] Oberly and Sumner also made the following conjecture:

Conjecture 1.2.8. [25] *If $k \geq 1$ and G is a $K_{1,k+2}$-free connected, locally k-connected graph of order at least 3, then G is hamiltonian.*

They were not entirely confident that this conjecture is true, but expressed confidence that a weaker alternative conjecture is true:

Conjecture 1.2.9. [25] *If $k \geq 1$ and G is a $K_{1,k+1}$-free connected, locally k-connected graph of order at least 3, then G is hamiltonian.*

Currently both conjectures are still open, although some progress has been made towards settling them. At a workshop hosted by the Banff International Research Station in August 2015, Susan van Aardt, Jean Dunbar, Marietjie Frick, Ortrud Oellermann and I considered a weaker connectivity condition: a graph G is *k-P_3-connected* if, for every pair u, v of non-adjacent vertices of G there exist k distinct $u - v$ paths of order 3 each. We proved the following result, which is somewhat weaker than Conjecture 1.2.9.

Theorem 1.2.10. [2] *If $k \geq 1$ and G is a connected, locally k-P_3-connected, $K_{1,k+2}$-free graph of order at least 3, then G is fully cycle extendable.*

I shall return to Oberly and Sumner's conjectures in Chapter 4. Oberly and Sumner [25] also speculated that connected LH graphs might be hamiltonian, but as they explain in a note at the end of their paper, it was pointed out to them even before their paper was published that this is not the case. The relationship between local and global hamiltonicity will be investigated in detail in Chapter 3.

Finally, Ryjáček [33] made a well-known conjecture relating to local connectedness:

Conjecture 1.2.11. [33] *Every LC graph is weakly pancyclic.*

This conjecture has been proven for several classes of LC graphs, such as maximal planar graphs and chordal graphs, and squares of graphs [33], but is seems difficult to settle for LC graphs in general [19], and even for LT and LH graphs.

Chapter 2

Locally Traceable Graphs

2.1 Introduction

Locally traceable graphs have received relatively little attention to date. In 1983 Pareek and Skupień [27] considered the traceability of LT and LH graphs. They posed a number of questions, one of which is related to LT graphs:

Question 1. [27] *Is 9 the smallest order of a connected nontraceable LT graph?*

In 1998 Asratian and Oksimets [7] considered graphs with hamiltonian balls, where a ball of radius r centered at a vertex v is the induced graph on vertices at a distance no greater than r from v (this includes v). A graph for which every ball of radius one is hamiltonian is simply a locally traceable graph. They proved the following two results (instead of using the hamiltonian ball terminology we use LT in the statement of these theorems).

Theorem 2.1.1. [7] *Let G be a connected LT graph of order $n \geq 3$. Then $|E(G)| \geq 2n - 3$.*

An outerplanar graph is a graph that can be embedded in the plane in such a way that every vertex borders the outer face. A graph is maximal outerplanar if no edge can be added while preserving outerplanarity.

Theorem 2.1.2. [7] *Let G be a connected LT graph of order $n \geq 3$. Then G is maximal outerplanar if and only if $|E(G)| = 2n - 3$.*

Since all maximal outerplanar graphs are hamiltonian, the next corollary follows readily:

Corollary 2.1.3. *Let G be a connected LT graph of order n that is not hamiltonian. Then $|E(G)| \geq 2n - 2$.*

In 2000 Alabdullatif [5] proved essentially the same results.

It is interesting to note that there is a close relationship between 2-trees and maximal outerplanar graphs. Markenzon et al. [22] proved the following result:

Theorem 2.1.4. [22] *A 2-tree G is a maximal outerplanar graph if and only if G is a SC 2-tree.*

Corollary 2.1.5. *A connected LT graph G of order n is a SC 2-tree if and only if $|E(G)| = 2n - 3$.*

However, not every 2-tree is LT and not every planar hamiltonian LT graph is a 2-tree - see Figure 2.1 for examples.

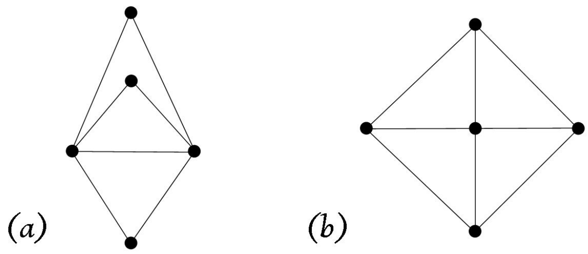

Figure 2.1: (a) a 2-tree that is not LT and (b) a planar LT graph that is not a 2-tree.

In Section 2.4 I show that the answer to Question 1 is "No, the smallest order is 10" and I present the 6 connected nontraceable LT graphs of order 10 that were found by means of a computer search. I also show that the maximum degree of nontraceable LT graphs is at least 6. I develop a technique that I call edge identification to construct nontraceable LT graphs, and use this technique to show that there are planar connected nontraceable LT graphs of all orders greater than 9. I show, moreover, that for every $n \geq 10$ there exists a connected nontraceable LT graph with maximum degree 7 and for every $n \geq 22$ there exists a connected nontraceable LT graph with maximum degree 6.

During a two-week workshop at Salt Rock in August 2013 Van Aardt, Frick, Oellerman and I [3] showed that the HCP for LT graphs with maximum degree at most 5 is fully solved (see Theorem 2.3.2 in Section 2.4). In Section 2.3 it will be shown that there exist connected nonhamiltonian LT graphs of order n with maximum degree 6 for every $n \geq 7$. It will also be shown that the HCP for LT graphs with maximum degree 6 is NP-complete.

2.2 Constructions and Preliminaries

We begin this section by defining a construction that will be extensively used in what follows.

Construction 2.2.1. (*Edge identification*) *Let G_1 and G_2 be two LT graphs such that $E(G_i)$ contains an edge u_iv_i so that there is a Hamilton path in $\langle N(u_i)\rangle$ that ends at v_i and a Hamilton path in $\langle N(v_i)\rangle$ that ends at u_i, $i = 1, 2$. Now create a larger graph G by identifying the edges u_1v_1 and u_2v_2 to a single edge uv (see Figure 2.2).* We say that G is obtained from G_1 and G_2 by identifying suitable edges.

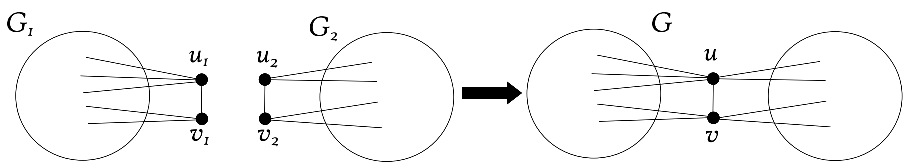

Figure 2.2: The edge identification procedure.

Theorem 2.2.2. *Let G_1 and G_2 be two LT graphs that satisfy the conditions of Construction 2.2.1. If G_1 and G_2 are combined by means of edge identification to create a graph G, then G is LT. If G is traceable, then both G_1 and G_2 are traceable.*

Proof. Let $u_iv_i \in E(G_i)$, $i = 1, 2$ be the two edges used in Construction 2.2.1 to form the edge uv in $E(G)$.

First suppose $w \in V(G) - \{u, v\}$. Since the neighbourhood of w is restricted to vertices that are either all in G_1 or all in G_2, $\langle N_G(w)\rangle$ is traceable.

Now suppose w is one of u and v, say u. Let Q_1v_1 be a Hamilton path in $\langle N_{G_1}(u_1)\rangle$ and let v_2Q_2 be a Hamilton path in $\langle N_{G_2}(u_2)\rangle$, where Q_1 and Q_2 are paths in G_1 and G_2, respectively. Then Q_1vQ_2 is a Hamilton path in $\langle N_G(u)\rangle$.

Using a similar argument, we can also find a Hamilton path in $\langle N_G(v)\rangle$. Hence G is LT.

Now assume P is a Hamilton path in G. If uv is an edge of P, then P is of the form Q_1uvQ_2 where Q_1uv and uvQ_2 are Hamilton paths of G_1 and G_2 respectively as illustrated in Figure 2.3 (a). If uv is not an edge of P, then P is of the form $Q_1uQ_2vQ_3$ where either Q_1uQ_2v is a Hamilton path of G_1 and uvQ_3 is a Hamilton path of G_2 or Q_1uvQ_3 is a Hamilton path of G_1 and uQ_2v is a Hamilton path of G_2 as illustrated by 2.3 (b) and (c) respectively.

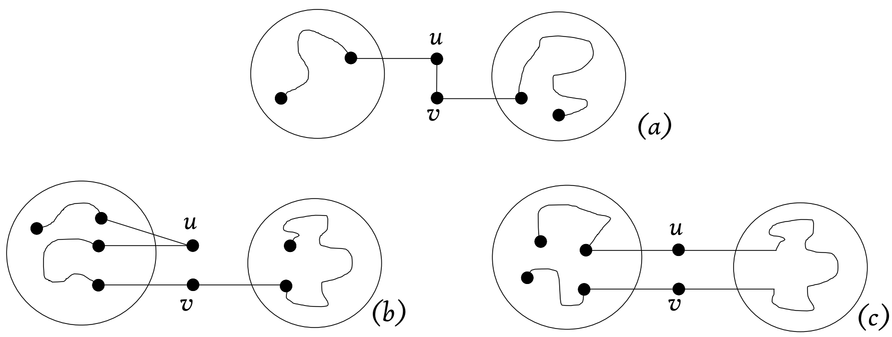

Figure 2.3: Edge identification preserves LT and nontraceable properties.

□

The following observation will be useful for selecting suitable edges to use in edge identification.

Observation 2.2.3. *Let v be a vertex of degree two in an LT graph. Then any edge incident with v is suitable for use in edge identification.*

This can easily be seen by noting that if $N(v) = \{u, w\}$, the edge uw is the Hamilton path of $\langle N(v)\rangle$, and since $d_{\langle N(u)\rangle}(v) = 1$, any Hamilton path of $\langle N(u)\rangle$ has v as an end vertex. In particular, if an LT graph G is combined with K_3 by means of edge identification to create a graph H, then the vertex $v \in V(K_3)$ that is not incident with the edge used in the procedure, has degree two. Hence any one of its incident edges is still suitable for use in edge identification.

The following observation is self-evident.

Observation 2.2.4. *If two planar LT graphs G_1 and G_2 are combined using edge identification to create graph G, then G is planar.*

The following variation on Construction 2.2.1 will also be needed.

Construction 2.2.5. (*Edge identification within a graph*) *Let G be an LT graph that contains disjoint edges u_iv_i, $i = 1, 2$, such that there is a Hamilton path in $\langle N(u_i)\rangle$ that ends at v_i and a Hamilton path in $\langle N(v_i)\rangle$ that ends at u_i. Furthermore, let $N(\{u_1, v_1\}) \cap N(\{u_2, v_2\}) = \emptyset$. Now create the graph G' by identifying the edges u_1v_1 and u_2v_2 to a single edge uv.* We say that G' is obtained from G by identifying suitable edges within G.

Theorem 2.2.6. *If a graph G' is constructed from an LT graph G by identifying suitable edges within G, then G' is also LT.*

Proof. Since $N(\{u_1, v_1\}) \cap N(\{u_2, v_2\}) = \emptyset$, the argument used in the proof of Theorem 2.2.2 applies here as well. □

When studying the hamiltonicity of *LT* graphs we will also need the following result.

Lemma 2.2.7. *Let G_1 and G_2 be two LT graphs, and let G be a graph obtained from G_1 and G_2 by identifying suitable edges. Then if G is hamiltonian, so are both G_1 and G_2.*

Proof. Let $u_iv_i \in E(G_i)$, $i = 1, 2$ be the edges that are identified to create the edge uv in G. Since $\{v, u\}$ is a cutset in G, it follows that no Hamilton cycle in G can include the edge vu. This implies that any Hamilton cycle in G has the form vQ_1uQ_2v where $v_1Q_1u_1$ is a Hamilton path in G_1 and $v_2Q_2u_2$ is a Hamilton path in G_2. Since $v_iu_i \in E(G_i)$ for $i = 1, 2$ it follows that each of G_1 and G_2 has a Hamilton cycle. □

2.3 Hamiltonicity of Locally Traceable Graphs

We start with a theorem by Van Aardt, Frick, Oellermann and de Wet [3] which fully solves the HCP for LT graphs with maxiumum degree at most 5. The first part of Section 2.3 (up to and including the proof of Theorem 2.3.2) has been published in [3].

Let $C = v_0v_1v_2 \ldots v_{t-1}v_0$ be a t-cycle in a graph G. If $i \neq j$ and $\{i, j\} \subseteq \{0, 1, \ldots, t-1\}$, then $v_i\overrightarrow{C}v_j$ and $v_i\overleftarrow{C}v_j$ denote, respectively, the paths $v_iv_{i+1} \ldots v_j$

and $v_iv_{i-1}\dots v_j$ (subscripts expressed modulo t). Let $C = v_0v_1\dots v_{t-1}v_1$ be a non-extendable cycle in a graph G. With reference to a given non-extendable cycle C, a vertex of G will be called a *cycle vertex* if it is on C, and an *off-cycle* vertex if it is in $V(G)-V(C)$. A cycle vertex that is adjacent to an off-cycle vertex will be called an *attachment vertex.* The following basic results on non-extendable cycles will be used frequently.

Lemma 2.3.1. [3] *Let $v_0v_1\dots v_{t-1}v_0$ be a non-extendable cycle C of length t in a graph G. Suppose v_i and v_j are two distinct attachment vertices of C that have a common off-cycle neighbour x. Then the following hold. (All subscripts are expressed modulo t.)*

1. $j \neq i+1$.
2. *Neither $v_{i+1}v_{j+1}$ nor $v_{i-1}v_{j-1}$ is in $E(G)$.*
3. *If $v_{i-1}v_{i+1} \in E(G)$, then neither $v_{j-1}v_i$ nor $v_{j+1}v_i$ is in $E(G)$.*
4. *If $j = i+2$ then v_{i+1} does not have two neighbours v_k, v_{k+1} on the path $v_{i+2}\dots v_i$.*

Proof. We prove each item by presenting an extension of C that would result if the given statement is assumed to be false. For (2) and (3) we only need to consider the first mentioned forbidden edge, due to symmetry.

1. $v_ixv_{i+1}\overrightarrow{C}v_i$.
2. $v_{i+1}v_{j+1}\overrightarrow{C}v_ixv_j\overleftarrow{C}v_{i+1}$.
3. $v_{j-1}v_ixv_j\overrightarrow{C}v_{i-1}v_{i+1}\overrightarrow{C}v_{j-1}$.
4. $v_kv_{i+1}v_{k+1}\overrightarrow{C}v_ixv_{i+2}\overrightarrow{C}v_k$.

□

It is well-known that for $k \geq 3$ the *wheel* W_k is obtained from a cycle $C = w_0w_1\dots w_{k-1}w_0$ by adding a new vertex w and joining it to every vertex of C. We call C the *rim* of the wheel, w its *centre* and edges of the type ww_i, $1 \leq i \leq k-1$, the *spokes* of the wheel. For $k \geq 3$, the *magwheel* M_k is the graph obtained from the wheel W_k by adding, for each edge e on the rim of W_k, a vertex v_e and joining it to the two ends of the edge e. Magwheels are examples of connected, nonhamiltonian LT graphs with $\delta = 2$. The magwheels with $\Delta \leq 5$ are depicted in Figure 2.4.

Since the graph $K_{1,1,3}$ is not LT, it follows from Theorem 1.2.1 that every connected, LT graph of order at least 3 and $\Delta \leq 4$ is hamiltonian. Moreover, if G is any graph with $\Delta = 5$ that satisfies conditions (b) or (c) of Theorem 1.2.5, then it is easily seen that G is not LT. However, magwheels are LT. Thus it follows from Theorems 1.2.1 and 1.2.5 that the magwheels M_3, M_4, M_5 are the only nonhamiltonian LT graphs with $\Delta \leq 5$. We now show that every connected LT graph with $\Delta = 5$ that is not a magwheel is fully cycle extendable.

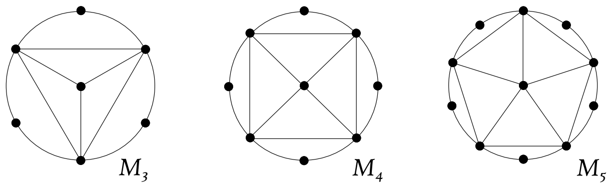

Figure 2.4: The graphs M_3, M_4 and M_5.

Theorem 2.3.2. [3] *Suppose G is a connected LT graph with $n(G) \geq 3$ and $\Delta(G) \leq 5$. Then G is fully cycle extendable if and only if $G \notin \{M_3, M_4, M_5\}$.*

Proof. It is easy to see that if $G \in \{M_3, M_4, M_5\}$, then G is not hamiltonian and hence not fully cycle extendable.

Now suppose that G is a connected locally traceable graph with $n(G) \geq 3$ and $\Delta(G) \leq 5$. Then $\delta(G) \geq 2$ and hence every vertex of G lies on a 3-cycle. If $n(G) = 3$ or 4, then G is obviously cycle extendable, so we assume $n(G) \geq 5$. Now suppose G has a non-extendable cycle $v_0 v_1 \ldots v_{t-1} v_0$ for some $t < n(G)$. Call the cycle C.

We first prove the following claim.

Claim 1. If v_i has an off-cycle neighbour x, then

(1) $v_{i-1}v_{i+1} \notin E(G)$,

(2) $N(v_i) = \{v_{i-2}, v_{i-1}, x, v_{i+1}, v_{i+2}\}$,

(3) x is adjacent to at least one of v_{i-2}, v_{i+2}.

Proof of Claim 1.

(1) Suppose $v_{i-1}v_{i+1} \in E(G)$. First suppose v_i has two distinct off-cycle neighbours x and y in $G - V(C)$. Then, since there are no edges from $\{v_{i-1}, v_{i+1}\}$ to $\{x, y\}$, we may assume, without loss of generality, that there is a 5-path $yxv_jv_{i+1}v_{i-1}$ in $\langle N(v_i) \rangle$, where v_j is necessarily a cycle vertex. Then, by Lemma 2.3.1(3), $j \notin \{i-2, i+2\}$. Hence, since $\Delta(G) \leq 5$, $N(v_j) = \{x, v_i, v_{i+1}, v_{j-1}, v_{j+1}\}$. By parts (1), (2) and (3) of Lemma 2.3.1, v_{j+1} is not adjacent to any of the vertices x, v_{i+1}, v_{j-1}. Also v_i is not adjacent to v_{j+1}, since $d(v_i) \leq 5$, so v_{j+1} is an isolated vertex in $\langle N(v_j) \rangle$ and hence $\langle N(v_j) \rangle$ is nontraceable, a contradiction.

Thus we may assume that v_i has only one off-cycle neighbour x, and x is adjacent to a vertex $v_j \in N(v_i)$. By Lemma 2.3.1(2) $j \neq i-2, i+2$. Also, by Lemma 2.3.1(1), $xv_{i-1}, xv_{i+1} \notin E(G)$.

If $d(v_i) = 4$, then, since $\langle N(v_i) \rangle$ is traceable, we may assume, without loss of generality, that $xv_jv_{i+1}v_{i-1}$ is a Hamilton path of $\langle N(v_i) \rangle$. Then, since $\Delta(G) \leq 5$, it follows that $N(v_j) = \{x, v_i, v_{i+1}, v_{j-1}, v_{j+1}\}$. Lemma 2.3.1(1) implies that $xv_{j-1}, xv_{j+1} \notin E(G)$. But $\langle N(v_j) \rangle$ is traceable, so v_{i+1} is adjacent to at least one of v_{j-1} and v_{j+1} and $v_{j-1}v_{j+1} \in E(G)$. This contradicts Lemma 2.3.1(3). Thus $d(v_i) = 5$.

Since v_i has only one off-cycle neighbour, there is a cycle vertex v_k such that $N(v_i) = \{v_{i-1}, v_{i+1}, x, v_j, v_k\}$. By symmetry we may assume that v_k lies on the path $v_{j+1}\overrightarrow{C}v_{i-2}$. Moreover, by Lemma 2.3.1(3), $k \neq j+1$. If $v_{i-1} \in N(v_j)$, then it follows from Lemma 2.3.1(3) that $v_{j-1}v_{j+1} \notin E(G)$. Then, by Lemma 2.3.1(2), v_{j-1} is not adjacent to v_{i-1} and hence not adjacent to any vertex in $N(v_j)$. Similarly, if $v_{i+1} \in N(v_j)$, then v_{j+1} is not adjacent to any vertex in $N(v_j)$. In either case, $\langle N(v_j) \rangle$ is not traceable. Hence v_j is not adjacent to either v_{i-1} or v_{i+1}. If v_k is adjacent to x, then a similar argument shows that v_k is not adjacent to either v_{i-1} or v_{i+1}. In this case $\langle N(v_i) \rangle$ has two distinct components which is not possible. Since $\langle N(v_i) \rangle$ is traceable it therefore follows that $v_kx \notin E(G)$ and that v_k is adjacent to v_j and one of v_{i-1} and v_{i+1}.

Suppose $k \notin \{j+2, i-2\}$. Then $N(v_j) = \{x, v_i, v_k, v_{j-1}, v_{j+1}\}$ and $N(v_k) =$

$\{v_i, v_j, v_{k-1}, v_{k+1}, v_s\}$, with s being either $i+1$ or $i-1$. Thus Lemma 2.3.1(1) and our assumption that $\Delta(G) \leq 5$, imply that there are no edges from the set $\{v_i, v_k, x\}$ to the set $\{v_{j-1}, v_{j+1}\}$, contradicting the fact that $\langle N(v_j)\rangle$ is traceable. Hence $k = i-2$ or $k = j+2$. In either case, since $\langle N(v_j)\rangle$ is traceable, $v_{j-1}v_{j+1} \in E(G)$. In the first case C extends to the cycle $v_{j-1}v_{j+1}\overrightarrow{C}v_k v_j x v_i v_{i-1} v_{i+1}\overrightarrow{C}v_{j-1}$. In the second case C extends to the cycle $v_{j-1}v_{j+1}v_j x v_i v_k\overrightarrow{C}v_{i-1}v_{i+1}\overrightarrow{C}v_{j-1}$.

(2) It follows from (1) above and Lemma 2.3.1(1) that the set $S = \{x, v_{i-1}, v_{i+1}\}$ is an independent set. Since $\langle N(v_i)\rangle$ is traceable, it follows that v_i has two cycle neighbours $v_j, v_k \notin S$. If v_j and v_k are consecutive vertices on C, then x is adjacent to only one of them and the other one is adjacent to both v_{i-1} and v_{i+1}. This contradicts Lemma 2.3.1(2). We may now assume that x is adjacent to v_j and that v_k lies on the path $v_{j+2}\overrightarrow{C}v_{i-2}$. Since $\Delta(G) = 5$, $N(v_i) = \{x, v_{i-1}, v_{i+1}, v_j, v_k\}$.

Suppose $j \neq i+2$. Since $\langle N(v_i)\rangle$ is traceable, v_j is adjacent to either v_{i-1} or v_{i+1}.

Case 1. $v_j v_{i-1} \in E(G)$.

In this case, $N(v_j) = \{x, v_i, v_{i-1}, v_{j-1}, v_{j+1}\}$. Our assumption that $j \neq i+2$ implies that v_{j-1} is not a neighbour of v_i. Furthermore, $x, v_{i-1}, v_{j+1} \notin N(v_{j-1})$ by parts 1, 2, and 3 of Lemma 2.3.1. Hence v_{j-1} has no neighbour in $N(v_j)$, so $\langle N(v_j)\rangle$ is not traceable.

Case 2. $v_j v_{i+1} \in E(G)$.

In this case $N(v_j) = \{x, v_i, v_{i+1}, v_{j-1}, v_{j+1}\}$. Now $v_{j+1} \notin N(v_i)$ and furthermore $x, v_{i+1}, v_{j-1} \notin N(v_{j+1})$ by parts 1, 2, 3 of Lemma 2.3.1. Hence again $\langle N(v_j)\rangle$ is not traceable.

Thus we have proved that in either case, $j = i+2$.

If v_k is adjacent to x, a symmetric argument proves that $k = i-2$ and this proves Claim 1(2) in this case.

Now assume that $k \neq i-2$ and $x \notin N(v_k)$. Since $\langle N(v_i)\rangle$ is traceable, both v_{i-1} and v_{i+1} are in $N(v_k)$. Hence $N(v_k) = \{v_{i-1}, v_i, v_{i+1},\ v_{k-1}, v_{k+1}\}$. Now

v_{k+1} is not a neighbour of v_{k-1}, since otherwise C can be extended to the cycle $v_{k-1}v_{k+1}\overrightarrow{C}v_{i-1}v_kv_{i+1}v_ixv_{i+2}\overrightarrow{C}v_{k-1}$. Also, v_{i-1} is not a neighbour of v_{k-1}, since otherwise C can be extended to the cycle $v_{k-1}v_{i-1}\overleftarrow{C}v_kv_{i+1}v_ixv_{i+2}\overrightarrow{C}v_{k-1}$. Also, by Lemma 2.3.1(4), $v_{i+1}v_{k-1} \notin E(G)$. So v_{k-1} has no neighbour in $N(v_k)$ and hence $\langle N(v_k)\rangle$ is not traceable. This proves that $k = i-2$. Thus we have proved (2).

(3) From the proof of (2) it follows, since $\langle N(v_i)\rangle$ is traceable, that x is adjacent with v_{i-2} or v_{i+2}. So (3) also holds.

Now suppose x is an off-cycle vertex that has a neighbour in C and consider the graph $G' = \langle V(C) \cup \{x\}\rangle$.

Suppose x is adjacent to every even-indexed cycle vertex. Then it follows from Lemma 2.3.1(1) that t is even, say $t = 2k$ and by Claim 1(1) and (2), no odd-indexed cycle vertex has an off-cycle neighbour. Since $\Delta(G) \le 5$, it follows that $k \le 5$ and no even-indexed cycle vertex has an off-cycle neighbour other than x. Hence $G = G'$. We also note that the odd-indexed cycle vertices are mutually nonadjacent, since otherwise G would be hamiltonian and cycle extendable. So in this case G is clearly isomorphic to a magwheel M_k for some $k \in \{3, 4, 5\}$.

Now assume that C has an even-indexed vertex that is not adjacent to x. Then, in view of Claim 1, we may assume without loss of generality that x is adjacent to both v_0 and v_2 but not to v_4.

Let

$$U_j = \{x\} \cup \{v_0, \ldots, v_j\},\ j = 1, \ldots, t-1.$$

We shall prove, by means of strong induction, that each of the following holds for $i = 2, 3, \ldots, \lfloor \frac{t-1}{2} \rfloor$.

(a) v_{2i} has a neighbour $b_i \in \{v_1, v_3, \ldots, v_{2i-3}\}$.

(b) G contains two $v_0 - v_{2i}$ paths $Q_{2i}(-b_i)$ and $Q_{2i}(-v_{2i-1})$ with vertex sets $U_{2i} - \{b_i\}$ and $U_{2i} - \{v_{2i-1}\}$, respectively.

(c) v_{2i-1} is not adjacent to any two consecutive vertices on the path $v_{2i}\overrightarrow{C}v_0$

(d) $N(v_{2i}) = \{b_i, v_{2i-2}, v_{2i-1}, v_{2i+1}, v_{2i+2}\}$.

Proof of the basis step ($i = 2$).

(a) Claim 1(2) implies that $N(v_2) = \{v_0, v_1, x, v_3, v_4\}$. By Lemma 2.3.1(1) and Claim 1(1), $I = \{x, v_1, v_3\}$ is an independent set in $\langle N(v_2)\rangle$. Since $\langle N(v_2)\rangle$ is traceable it follows that every vertex in $N(v_2) - I$ is adjacent to two vertices in I. But we have assumed that x is not a neighbour of v_4, so it follows that v_1 is a neighbour of v_4. Thus we put $b_2 = v_1$.

(b) The paths $Q_4(-b_2) = v_0xv_2v_3v_4$ and $Q_4(-v_3) = v_0xv_2v_1v_4$ are the desired $v_0 - v_4$ paths.

(c) Note that it follows from Claim 1(1) and the fact that $v_1v_4 \in E(G)$, that $t - 1 \neq 4$, so $t \geq 6$. Now suppose that v_3 has two consecutive neighbours v_j and v_{j+1} on the path $v_4\overrightarrow{C}v_0$. Then C can be extended to the cycle $v_{j+1}\overrightarrow{C}v_{t-1}Q_4(-v_3)v_5\overrightarrow{C}v_jv_3v_{j+1}$.

(d) We note that $\{v_1, v_2, v_3, v_5\} \subseteq N(v_4)$. By Lemma 2.3.1(4), v_1 does not have two consecutive neighbours on the path $v_4\overrightarrow{C}v_0$. By (c), the same is true for v_3. Since v_4 is a neighbour of both v_1 and v_3, it follows that v_5 is nonadjacent to both v_1 and v_3. We already know (from Claim 1(2)) that v_5 is also nonadjacent to v_2. Hence, since $\langle N(v_4)\rangle$ is traceable, v_4 has a fifth neighbour adjacent to v_5 which is a cycle vertex by Claim 1(1). Thus $N(v_4) = \{v_1, v_2, v_3, v_5, v_j\}$ where v_j is adjacent to v_5 and to at least one vertex in $\{v_1, v_3\}$.

Suppose $j > 6$. Then v_{j-1} and v_5 are distinct vertices. But $d(v_j) \leq 5$, so in this case v_j is adjacent to only one vertex in $\{v_1, v_3\}$. Call this vertex w. Then $N(v_j) = \{w, v_4, v_5, v_{j-1}, v_{j+1}\}$. We note that v_{j+1} is not adjacent to v_4, since $d(v_4) \leq 5$. Moreover, we have shown above that w does not have two consecutive neighbours on the path $v_4\overrightarrow{C}v_0$, so v_{j+1} is also nonadjacent to w. Furthermore, both v_5 and v_{j-1} are nonadjacent to v_{j+1}, since otherwise C extends to the respective cycles $v_{j+1}\overrightarrow{C}v_{t-1}Q_4(-w)wv_j\overleftarrow{C}v_5v_{j+1}$ and $v_{j+1}\overrightarrow{C}v_{t-1}Q_4(-w)wv_jv_5\overrightarrow{C}v_{j-1}v_{j+1}$. Thus v_{j+1} is not adjacent to any vertex in $N(v_j)$, contradicting the fact that $\langle N(v_j)\rangle$ is traceable. This proves that $j = 6$, and hence $N(v_4) = \{v_1, v_2, v_3, v_5, v_6\}$.

Thus the basis step is proved.

Proof of the induction step

Let r be an integer such that $4 \leq 2r \leq t-1$ and assume that (a), (b), (c) and (d) hold for every $i \in \{2, 3, \ldots, r-1\}$. We now prove that they also hold for $i = r$.

(a) Parts (a) and (d) of our induction hypothesis imply that there is a vertex $b_{r-1} \in \{v_1, v_3, \ldots, v_{2r-5}\}$ such that $N(v_{2r-2}) = \{b_{r-1}, v_{2r-4}, v_{2r-3}, v_{2r-1}, v_{2r}\}$ and also that $v_{2r-1}, v_{2r} \notin N(v_{2r-4})$. By part (a) of our induction hypothesis, $b_{r-1} \in \{v_1, \ldots, v_{2r-5}\}$. By part (c), neither v_{2r-3} nor v_{2r-1} is adjacent to b_{r-1}, and also, v_{2r-1} is not adjacent to v_{2r-3}. Hence, since $\langle N(v_{2r-2}) \rangle$ is traceable, v_{2r} is adjacent to a vertex $b_r \in \{v_{2r-3}, b_{r-1}\}$.

(b) Since b_r is either v_{2r-3} or b_{r-1}, part (b) of our induction hypothesis implies that there is a $v_0 - v_{2r-2}$ path $Q_{2r-2}(-b_r)$ with vertex set $U_{2r-2} - \{b_r\}$. Thus the desired $v_0 - v_{2r}$ paths are $Q_{2r}(-b_r) = Q_{2r-2}(-b_r)v_{2r-1}v_{2r}$ for $b_r = b_{r-1}$ and $Q_{2r}(-v_{2r-1}) = Q_{2r-2}(-b_r)b_r v_{2r}$ for $b_r = v_{2r-3}$.

(c) Suppose v_{2r-1} has two consecutive vertices v_j, v_{j+1} on the path $v_{2r}\overrightarrow{C}v_0$. Then C can be extended to the cycle $v_{j+1}\overrightarrow{C}v_{t-1}Q_{2r}(-v_{2r-1})v_{2r+1}\overrightarrow{C}v_j v_{2r-1}v_{j+1}$.

(d) We have shown that $\{b_r, v_{2r-2}, v_{2r-1}, v_{2r+1}\} \subseteq N(v_{2r})$. By parts (a) and (d) of our induction hypothesis, v_{2r+1} is not adjacent to v_{2r-2}. Moreover, it follows from (c) that v_{2r+1} is not adjacent to any neighbour of v_{2r} in $\{v_1, v_3, \ldots, v_{2r-1}\}$. Hence v_{2r+1} is not adjacent to any vertex in $\{b_r, v_{2r-2}, v_{2r-1}\}$. Since $\langle N(v_{2r}) \rangle$ is traceable, there is a cycle vertex v_j in $N(v_{2r})$ that is adjacent to v_{2r+1} and to at least one vertex in $\{b_r, v_{2r-1}\}$. Since v_j is adjacent to the two consecutive vertices v_{2r} and v_{2r+1}, it follows from Lemma 2.3.1(1) that v_j is indeed a cycle vertex. Moreover, by (c), $j \geq 2r+2$. Since $\Delta(G) \leq 5$,

$$N(v_{2r}) = \{b_r, v_{2r-2}, v_{2r-1}, v_{2r+1}, v_j, \}.$$

Suppose $j \neq 2r+2$. Then $N(v_j) = \{v_{2r}, v_{2r+1}, w_j, v_{j-1}, v_{j+1}\}$, where w_j is the neighbour of v_j in $\{b_r, v_{2r-1}\}$.

It follows from (c) that v_{j+1} is nonadjacent to w_j. Also, both v_{j-1} and v_{2r+1} are nonadjacent to v_{j+1}; otherwise (b) would imply that C can be extended to the respective cycles $v_{j+1}\overrightarrow{C}v_{t-1}Q_{2r}(-w_j)w_j v_j v_{2r+1}\overrightarrow{C}v_{j-1}v_{j+1}$ and

$v_{j+1}\overrightarrow{C}v_{t-1}Q_{2r}(-w_j)w_jv_j\overleftarrow{C}v_{2r+1}v_{j+1}$. Hence v_{j+1} has no neighbours in $N(v_j)$, contradicting the fact that $\langle N(v_j)\rangle$ is traceable. Hence $j = 2r + 2$ and thus

$$N(v_{2r}) = \{b_r, v_{2r-2}, v_{2r-1}, v_{2r+1}, v_{2r+2}\}.$$

This concludes the induction and proves that (a), (b), (c), (d) hold for every $i \in \{2, 3, \ldots, \lfloor (t-1)/2 \rfloor\}$.

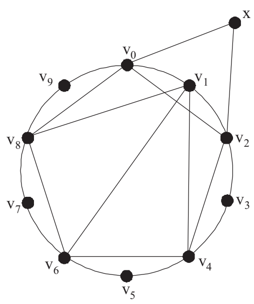

Figure 2.5: M_5, centered at v_1.

If t is odd, then it follows from (d) that $v_{t-1}v_1 \in E(G)$, contradicting Claim 1(1). Hence t is even, say $t = 2k$. We have shown that $N(v_{2k-2}) = \{v_{2k-4}, v_{2k-3}, v_{2k-1}, v_0, b_{k-1}\}$ where $b_{k-1} \in \{v_1, v_3, \ldots, v_{2k-5}\}$. Since $I = \{b_{k-1}, v_{2k-3}, v_{2k-1}\}$ is an independent set in $\langle N(v_{2k-2})\rangle$ and $\langle N(v_{2k-2})\rangle$ is traceable, v_0 has two neighbours in I. By Claim 1(2), v_0 is not adjacent to v_{2k-3}. Hence v_0 is adjacent to b_{k-1} and so $b_{k-1} = v_1$ by Claim 1.

But in the proof of (a) we showed that for each $i \in \{2, 3, \ldots, k-1\}$, the vertex b_i is either b_{i-1} or v_{2i-3}, so b_{i-1} lies on the path $v_0\overrightarrow{C}b_i$. Thus the fact that $b_{k-1} = v_1$ implies that $b_i = v_1$ for every $i \in \{1, 2, \ldots, k-1\}$.

Thus we have proved that v_{2i} is adjacent to v_1 for every $i \in \{0, 1, \ldots, k-1\}$. But then G is a magwheel with k spokes, centered at v_1, and $k \leq 5$ since $\Delta(G) \leq 5$. The case $k = 5$ is illustrated in Figure 2.5. □

Theorem 2.3.2 shows that there are only three nonhamiltonian connected *LT* graphs with maximum degree 5. For *LT* graphs with maximum degree 6 we now prove the following.

Theorem 2.3.3. *For any $n \geq 8$ there exists a nonhamiltonian planar connected LT graph G that has order n and maximum degree 6.*

Proof. Let G_7 be the graph M_3, depicted in Figure 2.4. For each $n \geq 8$, let G_n be the graph of order n obtained by combining G_{n-1} with a K_3 by means of edge identification, starting with the edge v_1v_2, and each time using one of the last edges added, choosing the edge such that the same vertex is never used more than twice, and specifically v_1 is only used once, as shown in Figure 2.6.

It follows from repeated application of Lemma 2.2.7 and Observation 2.2.4 that for $n \geq 7$, the graph G_n is a connected planar nonhamiltonian LT graph of order n and it is clear from Figure 2.6 that it has maximum degree 6 if $n \geq 8$. □

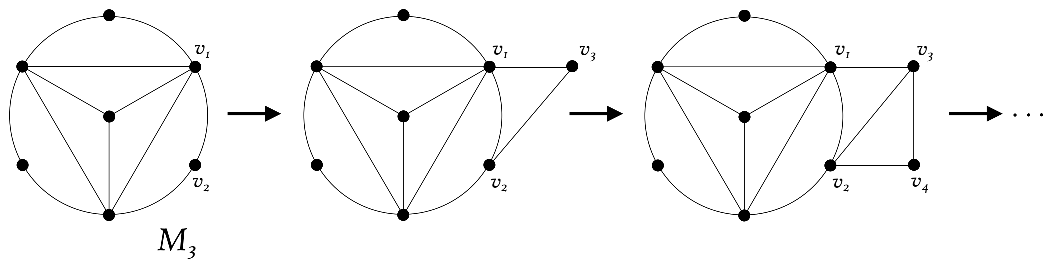

Figure 2.6: Constructing planar nonhamiltonian LT graphs with $\Delta(G) = 6$.

Corollary 2.1.3 says that if G is a nonhamiltonian connected LT graph of order n, then G has at least $2n - 2$ edges. Since the graph G_n defined in the proof of Theorem 2.3.3 has $2n-2$ edges, we now know that this bound is sharp. The following corollary follows easily from the proof of Theorem 2.3.3.

Corollary 2.3.4. *For each $n \geq 7$, there exists a nonhamiltonian connected LT graph of order n and size $2n - 2$.*

By Theorem 2.3.2, the HCP for LT graphs with maximum degree 5 is fully solved. I now show that for maximum degree 6 the problem is NP-complete. I shall need the following result by Akiyama, Nishizeki and Saito [4].

Theorem 2.3.5. [4] *The HCP is NP-complete for 2-connected cubic planar bipartite graphs.*

Theorem 2.3.6 has been submitted for publication in [35], although the proof presented there is somewhat more complex than the proof below.

Theorem 2.3.6. *The Hamilton Cycle Problem for planar LT graphs with maximum degree 6 is NP-complete.*

Proof. By to Theorem 2.3.5 the HCP for 2-connected cubic (i.e. 3-regular) planar bipartite graphs is NP-complete. Now consider any 2-connected planar cubic bipartite graph G'. We shall show that G' can be transformed in polynomial time to a planar LT graph G with $\Delta(G) = 6$ such that G is hamiltonian if and only if G' is hamiltonian.

Each vertex in G' will be represented by a triangle in G, and will be referred to as a node in G.

The edges in G' will be represented by a more complicated structure in G to ensure that G is LT and also that G is hamiltonian if and only if G' is hamiltonian. Consider the smallest of the magwheels, M_3, and the graph S in Figure 2.7. The graph M_3 and two copies of the graph S are combined by means of edge identification to create the graph B in Figure 2.8. This graph will be used in G to represent the edges in G', and will be referred to as a "border".

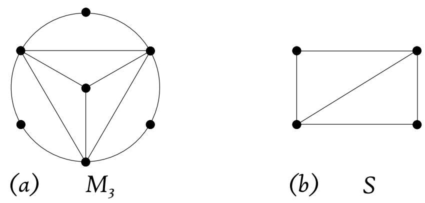

Figure 2.7: (a) The magwheel M_3 and (b) the graph S used in the proof of Theorem 2.3.6.

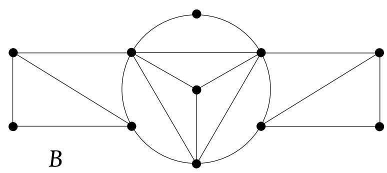

Figure 2.8: The border B used in the proof of Theorem 2.3.6.

Figure 2.9 shows how the graph G' is translated into graph G. In the figure, a vertex z_i in G' becomes a triangle Z_i in G and an edge e_j in G' becomes a border B_j in G. All the combinations of different components are done by means of edge identification, and it follows from Theorems 2.2.2 and 2.2.6 that the resulting graph is LT, and since G' is planar, so is G.

It remains to show that G is hamiltonian if and only if G' is hamiltonian. Figure 2.10 shows how a Hamilton cycle in G' translates to a Hamilton cycle in G. The

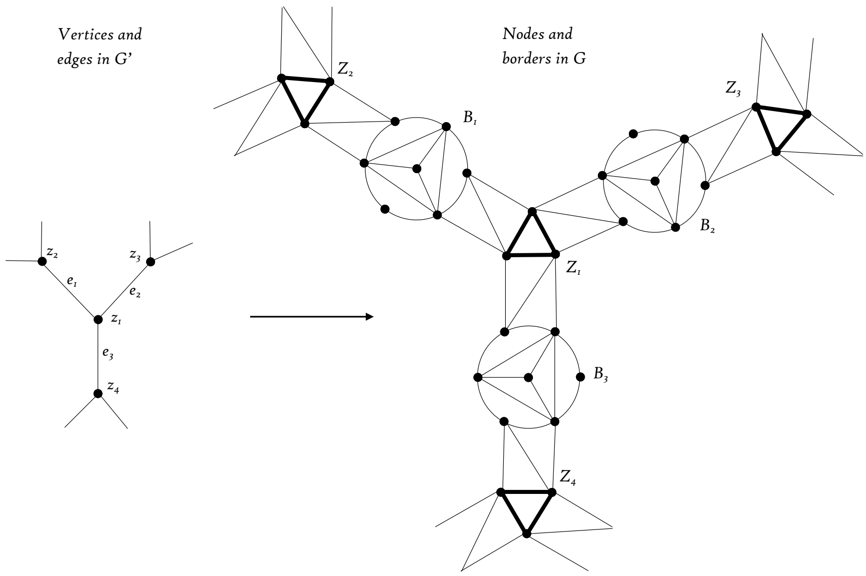

Figure 2.9: Translating graph G' into graph G in the proof of Theorem 2.3.6.

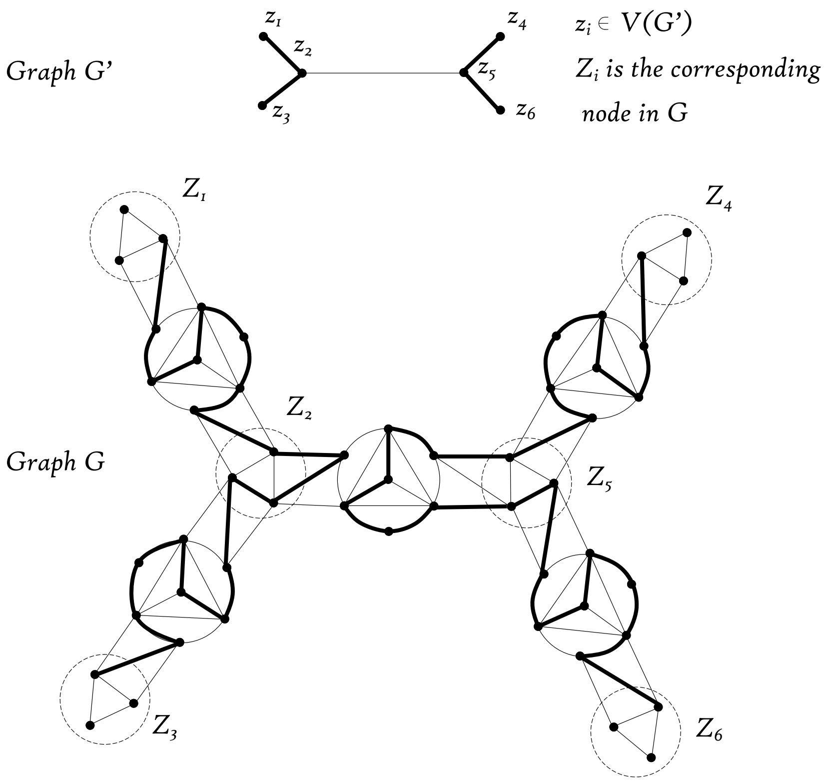

Figure 2.10: Translating a Hamilton cycle in G' into a Hamilton cycle in G in the proof of Theorem 2.3.6.

heavy lines in the figure represent edges that are part of the Hamilton cycles. Since each node has exactly three borders incident to it, all that is needed to show that G is not hamiltonian if G' is not hamiltonian is to show that a Hamilton cycle in

G can pass at most once through any given border between two nodes. Since the magwheel M_3 is nonhamiltonian, it follows that there does not exist a 2-path cover for M_3 for which the two pairs of end vertices are adjacent. Therefore there can be at most one path passing through a border from one node to another that includes all the vertices in the border.

□

Finally, I investigate the toughness of connected nonhamiltonian LT graphs. None of the small connected nonhamiltonian LT graphs depicted in this chapter is 1-tough, but it is possible to construct such graphs. I will make use of the fact that 3-connected cubic graphs are 1-tough, and that not all such graphs are hamiltonian [8].

Theorem 2.3.7. *For any $k \geq 6$ there exists a connected nonhamiltonian LT graph H_k with $\Delta(H_k) = k$ that is 1-tough.*

Proof. We use the same construction as in the proof of Theorem 2.3.6, but this time the graph G' is a nonhamiltonian 3-connected cubic graph. To see that the resulting graph G is 1-tough, we note that since G' is 1-tough, removing vertices only from the nodes of G does not result in more components than vertices removed (the nodes are cliques). The magwheel M_3 used to construct the borders in G is not 1-tough: if the three vertices of degree 5 (labeled say v_1, v_2, v_3) are removed, the result is a graph consisting of four isolated vertices. If v_1, v_2, v_3 are removed from a border in G, the resulting graph contains two isolated vertices, and the border no longer connects the two nodes incident to it in G. We will now proceed to remove the vertices in the position of v_1, v_2, v_3 from borders in G. Let G_m be the graph $G_{m-1} - \{v_{m,1}, v_{m,2}, v_{m,3}\} - \{u_{m,1}, u_{m,2}\}$, $m \geq 1$, where m is the number of borders that have been broken in this way, $v_{m,1}, v_{m,2}, v_{m,3}$ are the vertices in border m in the same relative position as v_1, v_2, v_3 that have been removed and $u_{m,1}$ and $u_{m,2}$ are the two vertices that have been isolated by the removal of $v_{m,1}, v_{m,2}, v_{m,3}$ (note that $G_0 = G$). Removing an edge in any graph increases the number of components by at most one, so removing the vertices $v_{m,1}, v_{m,2}, v_{m,3}$ from a border in G_{m-1} increases the number of components by at most 3 ($u_{m,1}$, $u_{m,2}$ and possibly the number of components of G_m increases by one). Since G' is 3-connected, at least 3 borders in

G have to be broken before G_m is disconnected. It follows that after two borders have been broken there are 4 isolated vertices and G_2 is still connected, and after m borders have been broken (by removing $3m$ vertices), the number of components in the resulting graph is at most $3+2+3+3+\cdots = 2+3(m-1) = 3m-1 < 3m$ and therefore G is 1-tough. To construct the graph H_k, where $k \geq 7$, simply connect G to a copy of K_{k-4} using edge identification on one of the edges that is incident to a vertex of degree 2 in a border in G. □

2.4 Traceability of Locally Traceable Graphs

The results in this section have been published in [34].

The first property of a connected nontraceable LT graph G I will investigate, is a lower bound for $\Delta(G)$.

Theorem 2.4.1. *If G is a connected nontraceable LT graph, then $\Delta(G) \geq 6$, and this bound is sharp.*

Proof. Since the graphs M_3, M_4 and M_5 in Figure 2.4 are traceable, it follows from Theorem 2.3.2 that $\Delta(G) \geq 6$ (a fully cycle extendable graph is hamiltonian, and therefore traceable). Four copies of the graph M_3 can be combined using edge identification to create the graph in Figure 2.11 with maximum degree 6. It is easy to see that this graph is nontraceable. Hence the bound is sharp. □

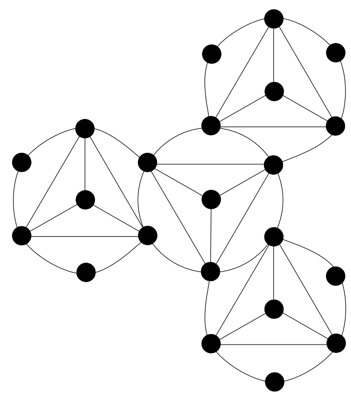

Figure 2.11: A connected nontraceable LT graph with maximum degree 6.

Next I answer Question 1 posed by Pareek and Skupień [27].

Theorem 2.4.2. *If G is a connected nontraceable LT graph, then $n(G) \geq 10$.*

Proof. By Theorem 2.4.1, G has a vertex w of degree k at least 6. Let $v_1v_2 \dots v_k$ be a Hamilton path of $\langle N(w) \rangle$, and let $X = \langle V(G) - N[w] \rangle$.

We make the following observations:

(i) $\langle N[w] \rangle$ is traceable from v_i to v_{i+1} (indices taken modulo k).

(ii) $\langle N[w] \rangle$ is traceable from v_1 and v_k to any vertex in $N[w]$.

(iii) Since $\langle N[w] \rangle$ is hamiltonian and G is nontraceable and LT, $n(X) \geq 2$.

(iv) Each component of X has at least two neighbours in $N(w)$.

(v) If $comp(X) \geq 2$, then X has at least three neighbours in $N(w)$.

Suppose $n(G) < 10$. Then it follows from Theorem 2.4.1 and (iii) above that $\Delta(G) = 6$, $n(X) = 2$ and $n(G) = 9$. Let $V(X) = \{x_1, x_2\}$. Since G is nontraceable, x_1 and x_2 are nonadjacent. Then by (ii) and (iv), no vertex in X can be adjacent to either v_1 or v_6. If x_1, say, is adjacent to both v_i and v_{i+1} (indices modulo 6), then $G - x_2$ is hamiltonian, and therefore G is traceable. If x_1 is adjacent to v_i and x_2 is adjacent to v_{i+1} (indices modulo 6), then by (i) G is traceable. Hence by (iv) and (v) we have a contradiction. □

A computer search of graphs of order 10 resulted in the 6 nontraceable LT graphs shown in Figure 2.12. The search was done by constructing all possible graphs of order 10 with maximum degree of either 6 or 7. The graphs were then tested for local traceability and traceability. Finally, graphs that were isomorphic to each other were eliminated from the list of graphs that were found. Since the search space is relatively small, it was feasible to do the search in Visual Basic in MicroSoft Excel. Note that all the graphs in Figure 2.12 have maximum degree 7. It is reasonably straightforward, although tedious, to prove analytically that every connected nontraceable LT graph of order 10 has maximum degree 7.

Theorem 2.4.3. *For any $k \geq 10$ there exists a connected planar nontraceable LT graph G that has order k and $\Delta(G) = 7$.*

Proof. Let G_0 be the graph LT10A, depicted in Figure 2.12 and redrawn as the first graph in Figure 2.13. For each $i \geq 1$, let G_i be the graph obtained by combining G_{i-1} with a K_3 by means of edge identification, starting with the edge v_1v_2, and after that each time using the edge between the vertices of degree two and three of

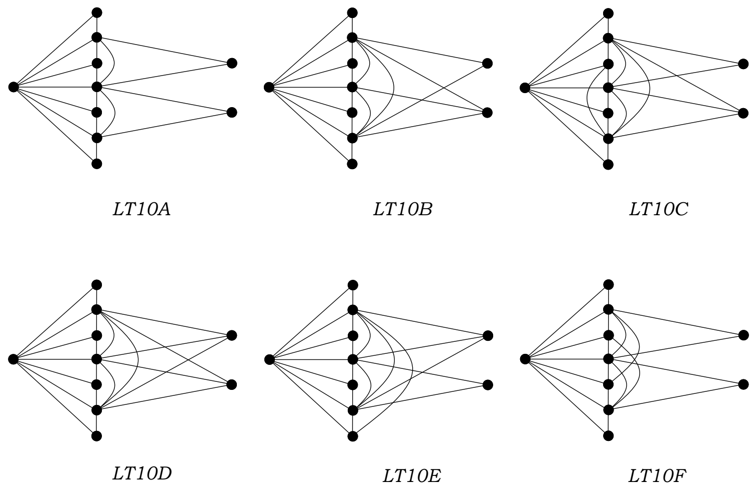

Figure 2.12: The nontraceable LT graphs of order 10.

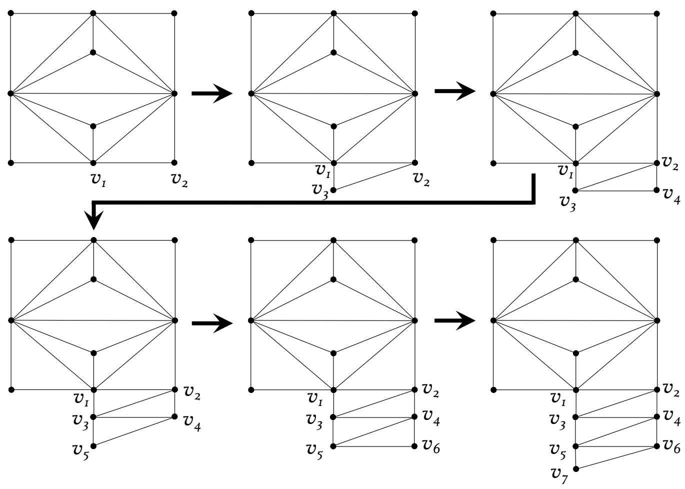

Figure 2.13: Constructing nontraceable LT graphs with $\Delta(G) = 7$.

the last added triangle, as shown in Figure 2.13. It follows from repeated application of Observation 2.2.4, that for $k \geq 10$, the graph G_{k-10} is a connected planar nontraceable LT graph of order k and it is clear from Figure 2.13 that it has maximum degree 7.

□

Note that the same procedure can be implemented using the graph in Figure 2.11 to create planar nontraceable LT graphs of any order greater than or equal to 22 with maximum degree 6.

Chapter 3

Locally Hamiltonian Graphs

3.1 Introduction

The notion of local hamiltonicity was introduced by Skupień [30] in 1965. He observed that any triangulation of a closed surface is LH. In particular, triangulations of the plane (maximal planar graphs) are LH. He also proved the following useful result.

Theorem 3.1.1. [29] *Suppose G is a connected LH graph of order $n \geq 3$. Then $|E(G)| \geq 3n - 6$. Moreover, $|E(G)| = 3n - 6$ if and only if G is a maximal planar graph.*

The following easy lemma was pointed out by Pareek and Skupień [27]:

Lemma 3.1.2. *If G is a connected LH graph of order n that is nonhamiltonian, then $\Delta(G) \leq n - 3$.*

In 1975 Goldner and Harary showed that the Goldner-Harary graph is the smallest maximal planar graph (and therefore the smallest connected planar LH graph) that is nonhamiltonian [18]. The Goldner-Harary graph has order 11 and size 27, and is shown in Figure 3.5. In 1983 Pareek and Skupień [27] extended this result to LH graphs:

Theorem 3.1.3. [27] *The smallest connected, nonhamiltonian LH graph has order 11.*

It follows from the next result by Chartrand and Pippert [11] that connected LH graphs are 3-connected.

Theorem 3.1.4. [11] *If a graph G is locally n-connected, $n \geq 1$, then every component of G is $(n+1)$-connected.*

The next result is fairly obvious.

Lemma 3.1.5. *Let G be an LH graph and let $v \in V(G)$. Then $\alpha(\langle N(v)\rangle) \leq d(v)/2$.*

There is a relationship between 3-trees and *LH* graphs similar to the one between 2-trees and *LT* graphs. Again, Markenzon et al. proved the relevant result:

Theorem 3.1.6. [22] *A graph G of order $n \geq 3$ is a SC-3-tree if and only if it is a chordal maximal planar graph.*

Corollary 3.1.7. *A connected LH graph G of order n is a SC 3-tree if and only if G is a chordal LH graph with $|E(G)| = 3n - 6$.*

In Section 3.2 I develop a technique called triangle identification that will be used extensively to manipulate and construct *LH* graphs with certain desired properties.

In Section 3.3 I investigate the global cycle properties of *LH* graphs with bounded maximum degree. The Goldner-Harary graph has maximum degree 8, and this led Pareek to speculate that every connected *LH* graph with maximum degree at most 7 is hamiltonian, and he published a proof for this [26]. However, I claim that his proof is not valid, and I explain the reasons for my claim. Nevertheless, it follows from Pareek's work and Theorem 3.3.1 that every connected *LH* graph with maximum degree 6 is hamiltonian. I show that for every $n \geq 11$ there exist connected nonhamiltonian *LH* graphs with maximum degree at most 9, but to date I have found only finitely many with maximum degree 8. I prove that the HCP for *LH* graphs with maximum degree 9 is NP-complete.

Pareek and Skupień [27] asked four questions regarding *LT* and *LH* graphs. The first question was addressed in Chapter 2 as Question 1. The other three questions will be addressed here:

Question 2. [27] *Is 14 the smallest order of a connected nontraceable LH graph?*

Question 3. [27] *Does there exist a nonhamiltonian connected LH graph that is regular?*

Question 4. [27] *Is K_4 the only regular LH graph that is not 4-connected?*

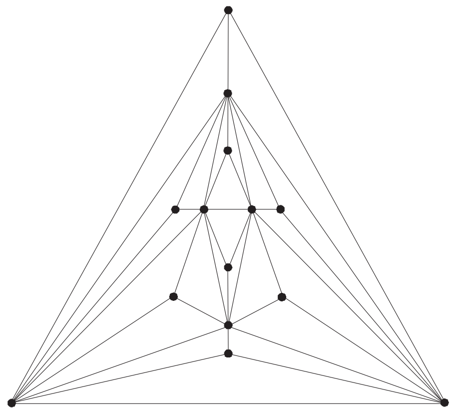

Figure 3.1: The Goodey graph (a connected nontraceable LH graph of order 14).

Figure 3.1 depicts a connected nontraceable LH graph of order 14. It was presented in 1972 as an example of a maximal planar nontraceable graph of smallest order by Goodey [17], who also proved that every maximal planar graph of order less than 14 is traceable.

In Section 3.4 I answer Question 2 in the affirmative by proving that there is no connected nontraceable LH graph of order less than 14. Using the triangle identification technique, I show that there are planar connected nontraceable LH graphs of every order greater than 13. I also show that there exist connected nontraceable LH graphs with minimum degree k for all $k \geq 3$.

In Section 3.5 I show by construction that the answer to Question 3 is positive. The constructed graphs have connectivity 3, so this answers Question 4 in the negative.

Entringer and MacKendrick [16] established an upper bound for $f(n)$, the largest integer such that every connected LH graph of order n contains a path of length $f(n)$. Their results imply that $\lim_{n\to\infty} f(n)/n = 0$. In Section 3.6 I show that if $p(n, \Delta)$ is the largest integer such that every connected planar LH graph of order n with maximum degree Δ contains a path of length $p(n, \Delta)$, then $\lim_{n\to\infty} p(n, \Delta)/n = 0$ for $\Delta \geq 11$.

3.2 Construction techniques for LH graphs

The following procedure will be used often to construct LH graphs with certain properties.

Construction 3.2.1. *For $i = 1, 2$, let G_i be an LH graph that contains a triangle X_i such that for each vertex $x \in V(X_i)$, there is a Hamilton cycle of $\langle N(x) \rangle$ that contains the edge $X_i - x$. Suppose $V(X_i) = \{u_i, v_i, w_i\}$, $i = 1, 2$. Now create a graph G of order $n(G_1)+n(G_2)-3$ by identifying the vertices u_i, $i = 1, 2$ to a single vertex u, and similarly the vertices v_i, $i = 1, 2$ to v and w_i, $i = 1, 2$ to w, while retaining all the edges present in the original two graphs (see Figure 3.2).* We say that G is obtained from G_1 and G_2 by identifying suitable triangles.

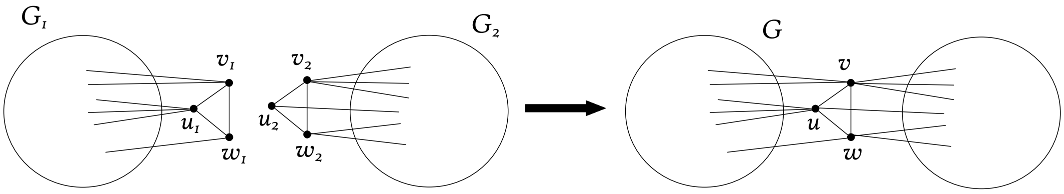

Figure 3.2: The triangle identification procedure.

Our next result shows that certain properties are retained when two graphs are combined by means of triangle identification.

Lemma 3.2.2. *Let G_1 and G_2 be two LH graphs, and let G be a graph obtained from G_1 and G_2 by identifying suitable triangles. Then*

(a) G is LH.

(b) If G_1 and G_2 are planar, then so is G.

(c) If G is hamiltonian, so are both G_1 and G_2.

(d) If G is traceable, so are both G_1 and G_2.

Proof. We use the notation defined in Construction 3.2.1.

(a) Let X be the triangle of G formed by identifying the vertices of X_1 and X_2 in Construction 3.2.1. Observe that if $y \in V(G_1 - X_1)$, then $N_G(y) = N_{G_1}(y)$, except for a possible label change of vertices in $N_{G_1}(y) \cap V(X_1)$ to the corresponding vertices in $V(X)$. Hence if $y \in V(G_1 - X_1)$, then $\langle N_G(y) \rangle$ is hamiltonian. The same is true for $y \in V(G_2 - X_2)$. Now suppose $y \in V(X)$, say $y = u$. Let $v_1 Q_1 w_1 v_1$ and $w_2 Q_2 v_2 w_2$ be Hamilton cycles of $\langle N_{G_1}(u_1) \rangle$ and $\langle N_{G_2}(u_2) \rangle$ respectively. Then $vQ_1 wQ_2 v$ is a Hamilton cycle of $\langle N_G(u) \rangle$. Using a similar argument, we can also find Hamilton cycles for $\langle N_G(v) \rangle$ and $\langle N_G(w) \rangle$.

(b) First we show that a separating triangle (a separating triangle is a triangle that does not border a face in a plane representation of the graph) is not suitable for use in triangle identification. Let v_1, v_2 and v_3 be the vertices of a separating triangle in G_1. Since LH graphs are 3-connected, each vertex in the separating triangle has neighbours both inside the triangle and outside the triangle. It follows that in $\langle N(v_1) \rangle$ the edge $v_2 v_3$ is a cut edge and is therefore not part of a Hamilton cycle in $\langle N(v_1) \rangle$. Therefore the triangle is not suitable for triangle identification.

Let X_1 and X_2 be the respective triangles of G_1 and G_2 that were used in the triangle identification procedure of Construction 3.2.1 to form the triangle X of G. Since G_1 and G_2 are planar, G_1 can be drawn such that the edges of X_1 border the outer face of G_1, and G_2 can be drawn such that the edges of X_2 border an inner face of G_2 in a plane representation. The triangle identification procedure then essentially draws $G_1 - X_1$ inside X and $G_2 - X_2$ outside X. Hence the resulting graph G is planar.

(c) First note that since $\{u, v, w\}$ is a cutset, it follows that no Hamilton cycle in G includes more than one edge between vertices in $\{u, v, w\}$. Figure 3.3 shows the only possible patterns that a Hamilton cycle in G can follow (the Hamilton cycle can include either one edge or no edges in $\langle \{u, v, w\} \rangle$). It follows that if G is hamiltonian, then so are both G_1 and G_2.

(d) Now suppose G is traceable. Since only vertices in $V(X)$ have neighbours in both G_1 and G_2, Figure 3.4 shows the possible patterns that a Hamilton path in G can follow. The Hamilton path in Figure 3.4(a) uses two edges of X, the ones in Figure 3.4(b)-(d) use only one edge of X and the ones in Figure 3.4(e)-(i) do not use any edge of X. In each case it is easily seen that each of G_1 and G_2 has a Hamilton path.

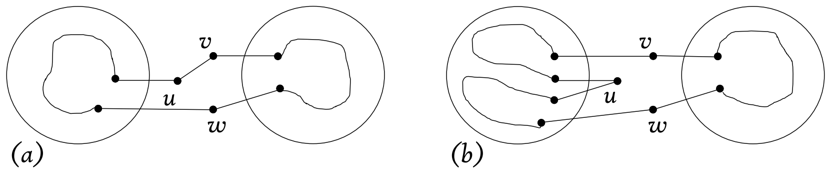

Figure 3.3: The possible Hamilton cycles through G.

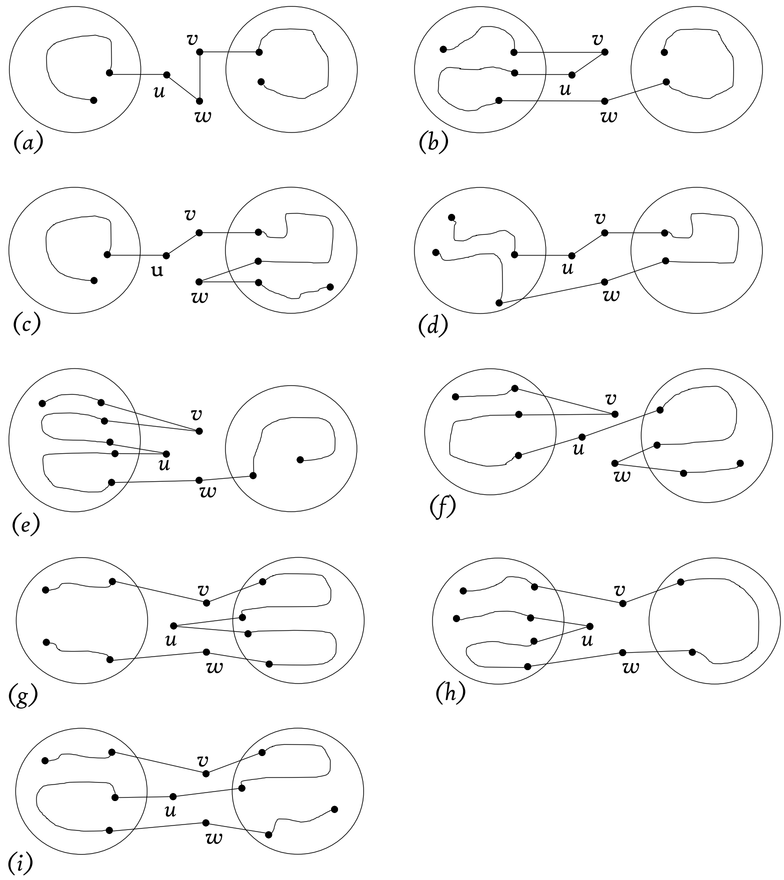

Figure 3.4: The possible Hamilton paths through G.

□

Note that it is possible to create a nonhamiltonian LH graph by using triangle identification to combine two hamiltonian LH graphs. In fact, it is possible to construct the Goldner-Harary graph using triangle identification and multiple copies of the graph K_4.

We will also need the following procedure, called *triangle identification within an LH graph*.

Construction 3.2.3. *Let G be an LH graph that contains disjoint triangles X_1 and X_2 such that $N(X_1) \cap N(X_2) = \emptyset$ and for each $x \in N(X_i)$ there is a Hamilton cycle* *of $\langle N(x)\rangle$ that contains the edge $X_i - x$, $i = 1, 2$. Let $V(X_i) = \{u_i, v_i, w_i\}, i = 1, 2$. Now create a graph G' of order $n(G) - 3$ from G by identifying u_i, $i = 1, 2$ to a single vertex u, and similarly the vertices v_i, $i = 1, 2$ to v and w_i, $i = 1, 2$ to w, while retaining all the edges present in the original graph.* We say that G' is obtained from G by identifying suitable triangles within G.

Lemma 3.2.4. *If G' is a graph obtained from an LH graph G by identifying two suitable triangles within G, then G' is LH.*

Proof. Let X_1 and X_2 be two suitable triangles in G. We use the same notation as in Construction 3.2.3. Note that the neighbourhood of a vertex $z \in V(G) - V(X_1) - V(X_2)$ is not changed by the construction (except for possible label changes, e.g., from u_i, $i = 1, 2$ to u), because $N(X_1) \cap N(X_2) = \emptyset$. Therefore, in G' only the neighbourhoods of u, v, w need to be considered. Let C_i be a Hamilton cycle of $\langle N_G(u_i)\rangle$ containing the edge $v_i w_i$, $i = 1, 2$. Then in G', the cycles C_1 and C_2 have only the edge vw in common, since $N_G(u_1) \cap N_G(u_2) = \emptyset$. Hence $C_1 - vw$ and $C_2 - vw$ can be combined to form a Hamilton cycle of $\langle N_{G'}(u)\rangle$. Similarly, we can prove that $\langle N_{G'}(v)\rangle$ and $\langle N_{G'}(w)\rangle$ are hamiltonian. Hence G' is LH. □

The final result in this section will be used in Section 3.6.

Lemma 3.2.5. *In an LH graph G, any vertex of degree 3 can be used three times in triangle identification, once in combination with each distinct subset of two of its three neighbours.*

Proof. Let $v_1 \in V(G)$ such that $N(v_1) = \{v_2, v_3, v_4\}$ and note that $\langle N[v_1]\rangle \cong K_4$. Since $d(v_1) = 3$, each triangle $\langle N[v_1] - v_i\rangle$, $i = 2, 3, 4$, is suitable for triangle identification. There are paths P_2, P_3 and P_4 in G such that the following are Hamilton cycles of $\langle N_G(v_i)\rangle$, $i = 1, 2, 3, 4$:
In $\langle N_G(v_1)\rangle$: $v_2v_3v_4v_2$
In $\langle N_G(v_2)\rangle$: $v_3v_1v_4P_2v_3$
In $\langle N_G(v_3)\rangle$: $v_2v_1v_4P_3v_2$
In $\langle N_G(v_4)\rangle$: $v_2v_1v_3P_4v_2$.

Let G_1 be an LH graph with a suitable triangle $X = \langle\{x_1, x_2, x_3\}\rangle$. For each $i = 1, 2, 3$, let Q_i be the path in the Hamilton cycle of $\langle N_{G_1}(x_i)\rangle$ between the end vertices of the edge $X - x_i$. Now use triangle identification to combine G with G_1 to form the graph H_1 by identifying the triangle $\langle\{v_1, v_2, v_3\}\rangle\rangle$ with the triangle $\langle\{x_1, x_2, x_3\}\rangle$. Let the identified vertices retain the labels v_1, v_2, v_3. By Lemma 3.2.2 (a), H_1 is LH and the following are Hamilton cycles of $\langle N_{H_1}(v_i)\rangle$, $i = 1, 2, 3, 4$:

In $\langle N_{H_1}(v_1)\rangle$: $C_{H_1,v_1} = v_2Q_1v_3v_4v_2$

In $\langle N_{H_1}(v_2)\rangle$: $C_{H_1,v_2} = v_3Q_2v_1v_4P_2v_3$

In $\langle N_{H_1}(v_3)\rangle$: $C_{H_1,v_3} = v_2Q_3v_1v_4P_3v_2$

In $\langle N_{H_1}(v_4)\rangle$: $C_{H_1,v_4} = v_2v_1v_3P_4v_2$.

The triangle $\langle\{v_1, v_2, v_4\}\rangle$ in H_1 is now suitable for triangle identification, since v_2v_4, v_1v_4, v_1v_2 are edges in C_{H_1,v_1}, C_{H_1,v_2}, C_{H_1,v_4} respectively.

Next, let G_2 be an LH graph with a suitable triangle $Y = \langle\{y_1, y_2, y_4\}\rangle$. For $i = 1, 2, 4$, let R_i be the path on the Hamilton cycle of $\langle N_{G_2}(y_i)\rangle$ between the end vertices of the edge $Y - y_i$. Now use triangle identification to combine H_1 with G_2 to form the graph H_2 by identifying the triangles $\langle\{v_1, v_2, v_4\}\rangle$ and $\langle\{y_1, y_2, y_4\}\rangle$. Let the identified vertices retain the lables v_1, v_2, v_4. By Lemma 3.2.2 (a), H_2 is LH and the following are Hamilton cycles of $\langle N_{H_2}(v_i)\rangle$, $i = 1, 2, 3, 4$:

In $\langle N_{H_2}(v_1)\rangle$: $C_{H_2,v_1} = v_2Q_1v_3v_4R_1v_2$

In $\langle N_{H_2}(v_2)\rangle$: $C_{H_2,v_2} = v_3Q_2v_1R_2v_4P_2v_3$

In $\langle N_{H_2}(v_3)\rangle$: $C_{H_2,v_3} = v_2Q_3v_1v_4P_3v_2$

In $\langle N_{H_2}(v_4)\rangle$: $C_{H_2,v_4} = v_2R_4v_1v_3P_4v_2$.

Since v_3v_4, v_1v_4 and v_1v_3 are edges in C_{H_2,v_4}, C_{H_2,v_3}, C_{H_2,v_4}, respectively, the triangle $\langle\{v_1, v_3, v_4,\}\rangle$ in H_2 is now suitable for triangle identification, so a third triangle identification, using this triangle, may be performed. □

Remark 3.2.6. *A given triangle may not be used more than once in triangle identification.*

To see that a triangle with vertices x_1, x_2 and x_3 in an LH graph G_1 can only be used once in triangle identification to combine G_1 with an LH graph G_2, note that before triangle identification the edge x_2x_3 is part of a Hamilton cycle in $\langle N_{G_1}(x_1)\rangle$. After triangle identification, the edge x_2x_3 is replaced in the Hamilton cycle in $\langle N_G(x_1)\rangle$ by a path with vertices that originated from G_2. The same constraint does not apply to vertices.

3.3 Global Cycle Properties of Locally Hamiltonian Graphs with Bounded Maximum Degree

A computer search for order 11 LH graphs found the four graphs in Figure 3.5. Graph G11A is the Goldner-Harary graph and graph G11B was first found by one of my supervisors (Frick). Note that $G11A$ is a maximal planar graph and has size 27, while the other three graphs have size 30 and are therefore not planar. Also note that all four graphs have maximum degree 8.

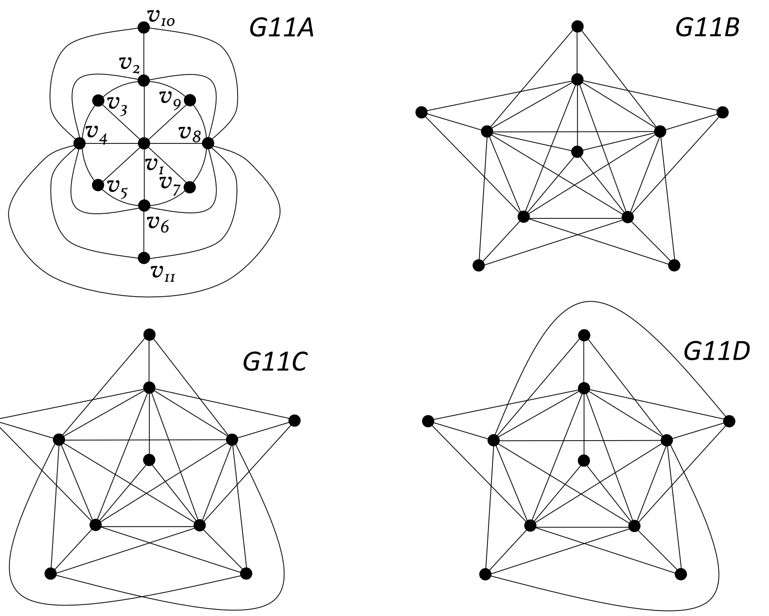

Figure 3.5: Nonhamiltonian LH graphs of order 11.

In 1983 Pareek [26] published a paper claiming that every connected LH graph with maximum degree less than 8 is hamiltonian. However, the proof in his paper omits several special cases, and some of the claims that he makes on which he bases further results are false.

Pareek's proof will not be set out in detail. Rather, I will focus on the main reasons why I believe it is not valid (this discussion has also been submitted for publication in [35]). Pareek considers a longest cycle $C = v_1v_2 \ldots v_tv_1$ in an LH graph G with $\Delta(G) \leq 7$. He shows that if G is not hamiltonian, then C contains a vertex v_1 of degree at least 7 that has 6 neighbours on C and one neighbour x in $G - V(C)$. Let $N(v_1) = \{x, v_2, v_i, v_j, v_k, v_l, v_t\}$. Since $\langle N(v_1)\rangle$ is hamiltonian, x has two neighbours in $N(v_1)$, say v_i and v_k. It suffices to consider the following three cases (Figure 3.6). The possibility that a graph may belong to both Case 1 and Case 2 is not explicitly considered, but does not affect the logic of the argument.

Case 1. $v_{k+1} \in N(v_1)$.

Case 2. $v_{k-1} \in N(v_1)$.

Case 3. $N(v_1) \cap \{v_{i-1}, v_{i+1}, v_{k-1}, v_{k+1}\} = \emptyset$.

Since $\langle N(v_k) \rangle$ is hamiltonian, v_k and x have a common neighbour $v_p \neq v_1$ on C.

I agree up to this point. But then Pareek claims that Case 3 converts to either Case 1 or Case 2 and I do not agree with that. Pareek argues that in Case 3, the

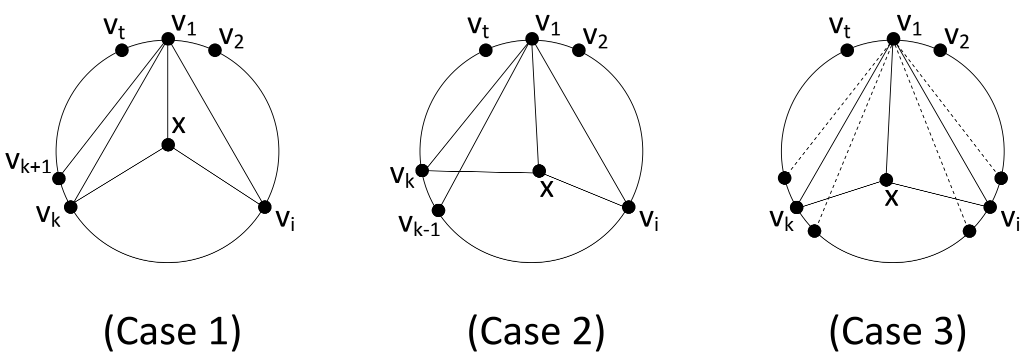

Figure 3.6: The three cases used in Pareek's proof.

fact that the neighbourhoods of v_1, v_i, v_k, v_j, v_l and v_p induce hamiltonian graphs implies that $d_C(v_p) = 6$ and that v_p has a neighbour in $\{v_{k-1}, v_{k+1}\}$. By relabelling the vertices so that v_p becomes v_1, it would then follow that this case converts to either Case 1 or Case 2. However, Figure 3.7 (a) shows an example of such a situation where the neighbourhoods of v_1, v_i, v_k, v_j, v_l and v_p induce hamiltonian graphs, but neither v_k nor v_i has consecutive neighbours on C. This case does therefore not convert to Case 1 or Case 2. (I have illustrated the case where $v_p = v_i$, as this leads to the simplest example, but even if v_p and v_i are distinct, the same kind of counterexample is possible.)

The next step in Pareek's proof is to show that if Case 1 occurs, then so does Case 2. I do not agree with this either. The graph in Figure 3.7 (b) is a counterexample: the neighbourhoods of v_1, v_i and v_k induce hamiltonian graphs, but Case 2 does not occur (it is also possible to find Hamilton cycles in the graphs induced by the neighbourhoods of the unlabeled vertices in the figure, but for the sake of clarity

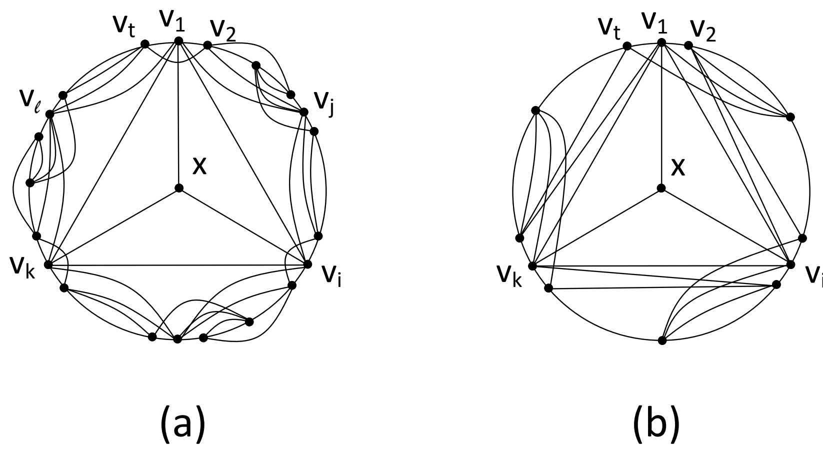

Figure 3.7: Counterexamples to Pareek's Claims.

these are not shown).

Pareek's final step is to show that Case 2 is not possible. However, he omits some of the possible subcases of Case 2, but more seriously, the proof fails if $k < p < t$.

I therefore regard the problem as to whether there exists a nonhamiltonian connected LH graph with maximum degree 7 as unsolved. Nevertheless, it follows from the correct part of Pareek's proof that every connected LH graph with maximum degree at least 6 is hamiltonian. Moreover, at the mentioned Salt Rock workshop, we adapted the technique that Pareek had used to prove the following (this was published as [3]).

Theorem 3.3.1. [3] *Let G be a connected LH graph with $n(G) \geq 3$ and $\Delta(G) \leq 6$. Then G is fully cycle extendable.*

Proof. Since G is locally hamiltonian, every vertex lies on a 3-cycle. It suffices thus to show that every cycle is extendable. Assume, to the contrary, that there is a cycle $C = v_0v_1 \dots v_{t-1}v_0$ of length $t < n(G)$ that is not extendable. Since G is connected, some vertex of C, say v_0, has an off-cycle neighbour x. Since $\langle N(v_0) \rangle$ contains a Hamilton cycle H_{v_0}, it contains two $x - C$ paths that are disjoint except for x. Let v_j and v_k be the first cycle vertices on the respective paths where $j < k$. Then there are off-cycle vertices $x_j, x_k \in N(v_0)$ (at least one of which is x, since $deg\ v_0 \leq 6$.) such that x_j is adjacent to v_j and x_k is adjacent to v_k. By Lemma 2.3.1(1), $j, k \notin \{1, t-1\}$.

First, suppose v_1, v_{t-1}, v_j, v_k are the only neighbours of v_0 on C. Then $v_kv_{t-1}v_1v_j$ or $v_kv_1v_{t-1}v_j$ is a subpath of H_{v_0}. Assume the former. (The latter case can be handled similary.) By Lemma 2.3.1(3), $j \neq 2$ and $k \neq t-2$.

It follows from Lemma 2.3.1(1) and (3) that $I_k = \{x_k, v_{k-1}, v_{k+1}\}$ is an independent set in $\langle N(v_k)\rangle$. Hence, since $\langle N(v_k)\rangle$ has a Hamilton cycle, $|N(v_k)| = 6$ and every vertex in $N(v_k) - I_k$ is adjacent to two vertices in I_k. But then v_0 is adjacent to at least one of v_{k-1} and v_{k+1}, contradicting Lemma 2.3.1(3). Hence v_0 has exactly five neighbours on C. In fact, this proves that every attachment vertex of C has exactly 5 cycle neighbours and one off-cycle neighbour.

Thus we may assume that $N(v_0) = \{x, v_1, v_{t-1}, v_j, v_k, v_q\}$, where $j < k$ and v_jxv_k is a path on a Hamilton cycle H_{v_0} of $\langle N(v_0)\rangle$ and v_q is another cycle neighbour of v_0. Thus we may assume without loss of generality that H_{v_0} contains the edge v_jv_1 or v_jv_{t-1}. Then it follows from Lemma 2.3.1(3) that $v_{j-1}v_{j+1} \notin E(G)$. Hence $I_j = \{x, v_{j-1}, v_{j+1}\}$ is an independent set in $\langle N(v_j)\rangle$. Hence every vertex in $N(v_j)-I_j$ is adjacent to at least two vertices in I_j. But by Lemma 2.3.1(2), $v_{t-1}v_{j-1} \notin E(G)$ so $v_{t-1} \notin N(v_j)$. Hence $v_1 \in N(v_j)$. But since $v_1v_{j+1} \notin E(G)$ it follows that $v_1 = v_{j-1}$, i.e. $j = 2$.

By Lemma 2.3.1(3), $v_{t-1}v_1 \notin E(G)$, so v_k is adjacent to v_1 or v_{t-1}. Thus a similar argument as above shows that $k = t-2$. Since the path $v_{t-2}xv_2$ lies on H_{v_0}, the fact that $v_{t-1}v_1 \notin E(G)$ implies that $v_1v_qv_{t-1}$ also lies on H_{v_0}. Hence $3 < q < t-3$ by Lemma 2.3.1(2). We observe that $v_{q-1}v_{q+1} \notin E(G)$, since otherwise, $v_{q-1}v_{q+1}\overrightarrow{C}v_{t-1}v_qv_1v_0xv_2\overrightarrow{C}v_{q-1}$ is a $(t+1)$-cycle that contains the vertices of C, a contradiction. But by Lemma 2.3.1(4), neither v_{q-1} nor v_{q+1} is adjacent to either v_{t-1} or v_1. Hence $\{v_1, v_{t-1}, v_{q-1}, v_{q+1}\}$ is an independent set in $\langle N(v_q)\rangle$. But, since $|N(v_q)| \leq 6$ it follows that $\langle N(v_q)\rangle$ is nonhamiltonian. This contradiction produces the desired result. □

Theorem 3.3.1 extends the result of Altshuler [6] that any 6-regular triangulation of the torus is hamiltonian.

In order to prove the next theorem we will need a planar LH graph of any order $n \geq 4$ with maximum degree at most 6 that contains a triangle with vertices u_1, u_2 and u_3 of degrees 3, 4 and 5 respectively. Observation 3.3.2 shows how to construct such a graph.

Observation 3.3.2. *There exists a planar LH graph G of order n for every* $n \geq 4$ *such that* $\Delta(G) \leq 6$ *and G contains a triangle whose vertices have degrees 3, 4 and 5.*

Proof. Such a graph can be constructed in the following manner: start with K_4 drawn in a plane representation. Attach an additional vertex to the three outer vertices in K_4 to create graph G_5. Keep repeating this procedure (add an additional vertex by connecting it to the three outer vertices in G_i). The procedure essentially starts off with K_4, which is LH, and in each step uses triangle identification to combine G_i with K_4, so it is clear that the new graph G_{i+1} is also LH. Moreover, by drawing the graph in each step so that edges between the last three vertices added border the outer plane, the maximum degree can be limited to six, and the last three vertices added have degrees 5, 4 and 3, respectively. See Figure 3.8. □

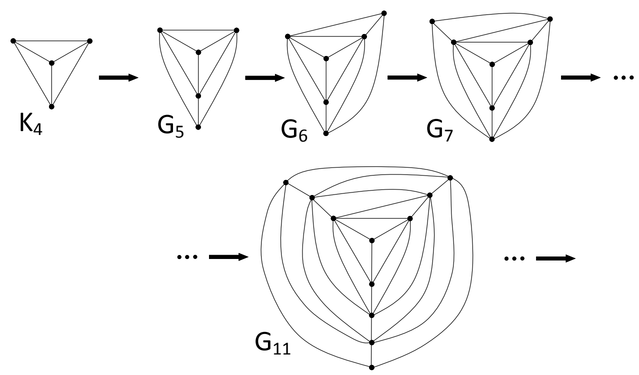

Figure 3.8: Constructing a planar LH graph with maximum degree 6.

Theorem 3.3.3. *For every* $n \geq 11$ *there exists a connected planar nonhamiltonian LH graph G with* $\Delta(G) \leq 9$.

Proof. For any $k \geq 4$, Let H_k be a planar LH graph of order k with $\Delta(H_k) \leq 6$ such that H_k contains a triangle with vertices u_1, u_2 and u_3 of degrees 3, 4 and 5 respectively. Using vertices with low degrees in triangle identification limits the degrees of the resulting identified vertices. Now combine combine H_k with the graph $G11A$ in Figure 3.5 using triangle identification by identifying u_1 with v_1, u_2 with v_2 and u_3 with v_3. Then the resulting graph G is a connected graph with $\Delta(G) = 9$ and $n(G) = 11 + k - 3$ and, by Lemma 3.2.2 (b) and (c), G is both planar and nonhamiltonian. □

I have found nonhamiltonian connected LH graphs with maximum degree 8 and order 11, 13, 14, 15, and as large as 34, but I do not know whether there are infinitely many. The following theorem shows that there are none of order 12. The proof is long and uninteresting, and can be found in Appendix 1. The result will be needed to prove Theorem 4.2.7.

Theorem 3.3.4. *Let G be a connected nonhamiltonian LH graph of order $n = 12$. Then $\Delta(G) = 9$.*

Chvátal [12] and Wigderson [36] independently proved that the Hamilton Cycle Problem for maximal planar graphs is NP-complete. Although neither author was interested in the minimum value of the maximum degree for which this is true, it is straightforward to manipulate the construction Chvátal used to show that the theorem holds for a maximum degree as low as 12. However, I shall make a further improvement for LH graphs (that is, if we drop the requirement that the graph be planar). A weaker version of Theorem 3.3.5 has been submitted for publication in [35] (The Hamilton Cycle Problem for LH graphs with maximum degree 10 is NP-complete).

Theorem 3.3.5. *The Hamilton Cycle Problem for LH graphs with maximum degree 9 is NP-complete.*

Proof. Starting with a cubic graph G', we will construct a connected LH graph G with $\Delta(G) = 9$ such that G is hamiltonian if and only if G' is hamiltonian.

Each vertex in G' is replaced by a copy of a K_4 graph in G, and will be referred to as a node in G.

The edges will be replaced by a more complex structure, both to ensure local hamiltonicity and to ensure that G is hamiltonian if and only if G' is hamiltonian. Consider the nonhamiltonian LH Goldner-Harary graph H in Figure 3.9 (a) and the LH graph D in Figure 3.9 (b). We use triangle identification to combine H with two copies of D in the following way: using the first copy of D, identify v_1 and x_1, v_2 and x_2, and v_3 and x_3, and using the second copy of D, identify u_1 and x_1, u_2 and x_2, and u_3 and x_3. This yields the graph F_i in Figure 3.10, which is LH and nonhamiltonian.

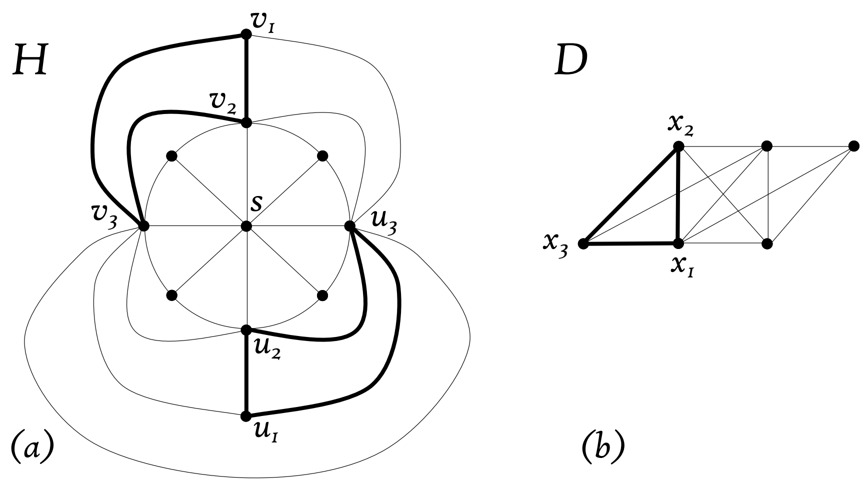

Figure 3.9: (a) The Goldner-Harary graph H and (b) the graph D used in the proof of Theorem 3.3.5.

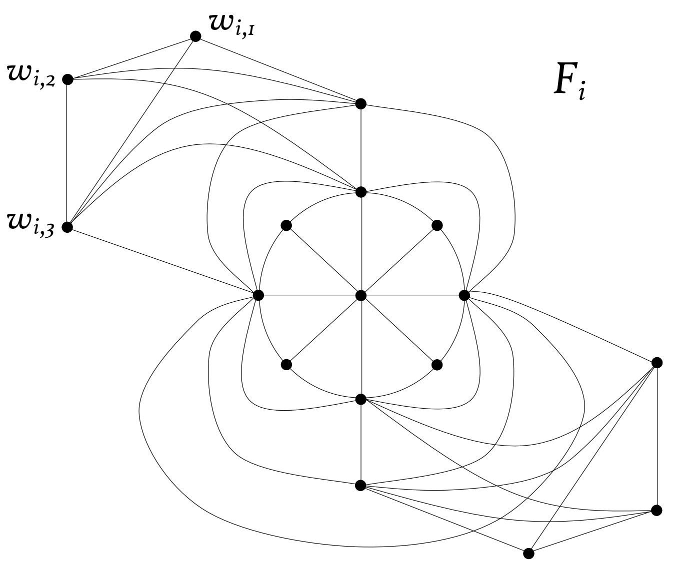

Figure 3.10: The graph F_i used in the proof of Theorem 3.3.5.

The graphs F_i will be used to connect the nodes in G and will be referred to as "borders". Thus each edge in G' will be replaced by one border. The borders are connected to the nodes by means of triangle identification. Let the vertices in a node in G be y_1, y_2, y_3, y_4 and let the vertices in F_i be as shown in Figure 3.10. Since each vertex in G' has degree three, each node in G is attached to three copies of F_i. We identify the vertices as shown in Table 3.1. We use the graphs F_1, F_2 and F_3 for illustrative purposes. See Figure 3.11 (the heavy lines in G represent edges belonging to the nodes).

Vertex in node	Vertex in F_i
y_1	$w_{1,1}$
y_2	$w_{1,2}$
y_3	$w_{1,3}$
y_2	$w_{2,2}$
y_3	$w_{2,1}$
y_4	$w_{2,3}$
y_1	$w_{3,3}$
y_2	$w_{3,2}$
y_4	$w_{3,1}$

Table 3.1: Vertices identified in the proof of Theorem 3.3.5.

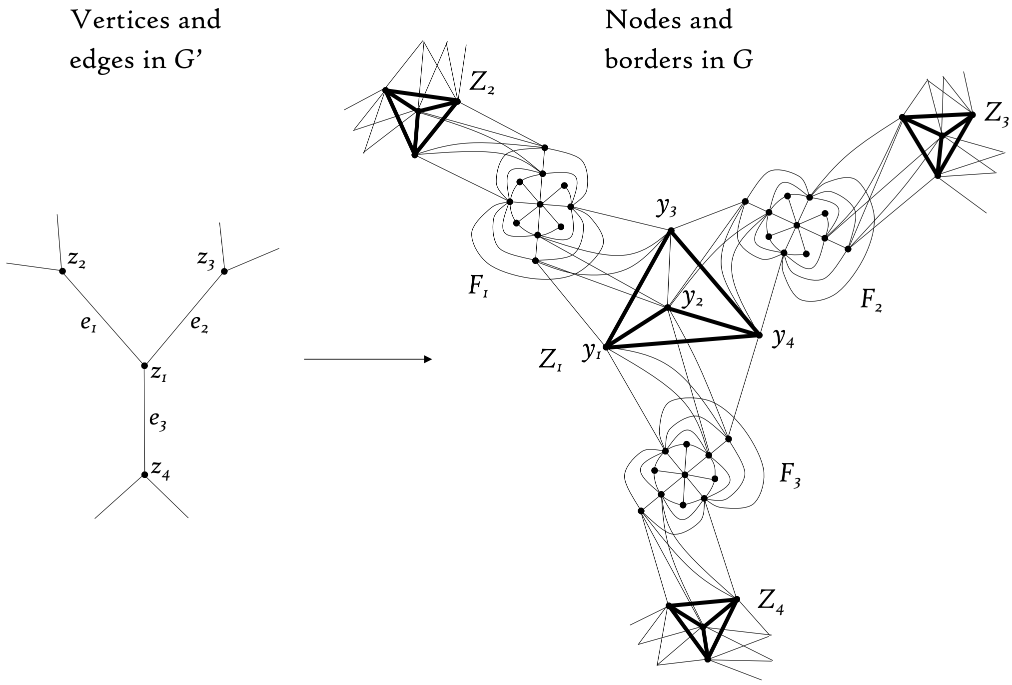

Figure 3.11: Converting the graph G' to G.

Checking the degrees of the vertices that have been identified shows that $\Delta(G) = 9$ and by Lemmas 3.2.2 (a), 3.2.4 and 3.2.5, G is LH.

We still have to show that G is hamiltonian if and only if G' is. Figure 3.12 shows how a Hamilton cycle in G' translates to a Hamilton cycle in G (the heavy lines represent paths in the Hamilton cycle).

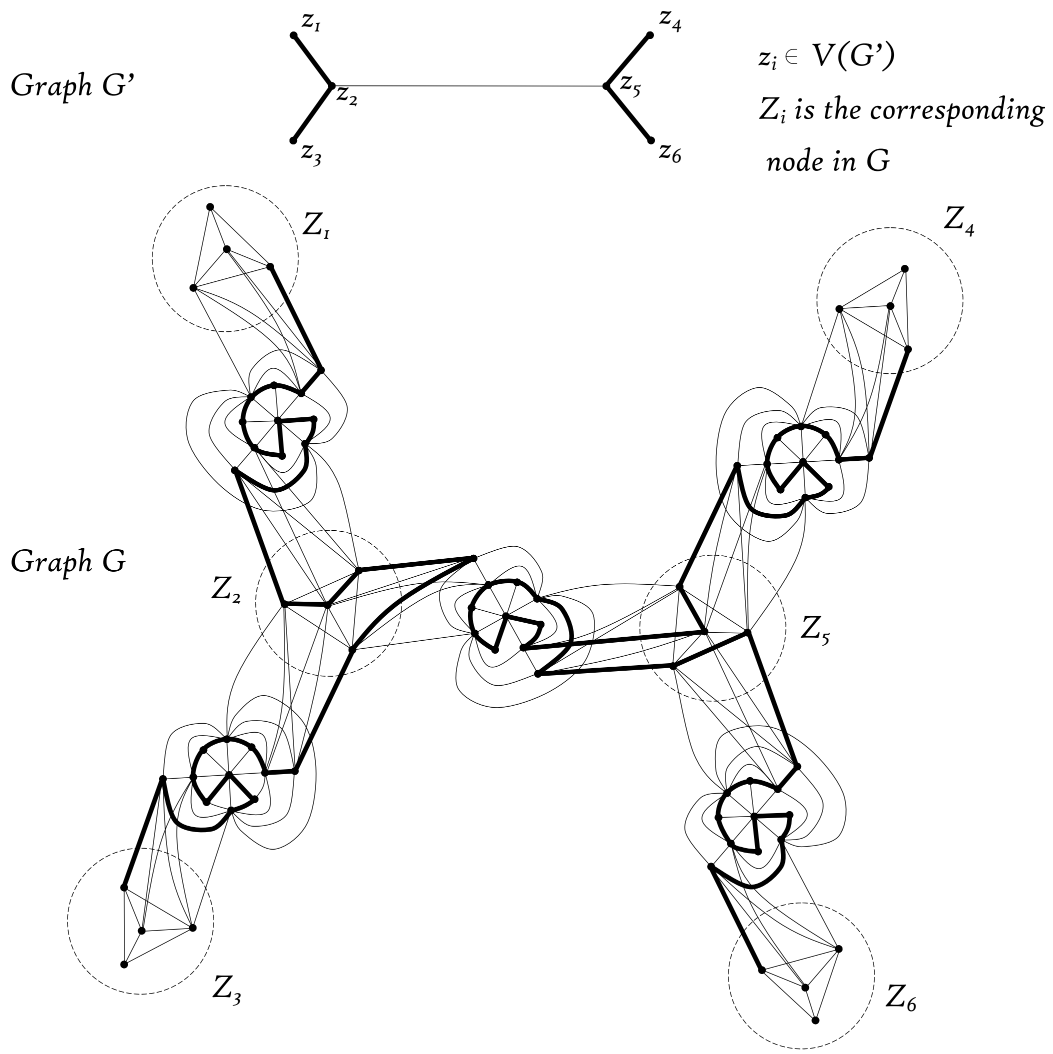

Figure 3.12: Translating a Hamilton cycle from G' to G.

Consider a copy of H in a border of G that connects two nodes, say Z_1 and Z_2. Assume that the edges between H and Z_1 are incident with vertices in $\{u_1, u_2, u_3\}$, and the edges between H and Z_2 are incident with vertices in $\{v_1, v_2, v_3\}$ (as labelled in Figure 1(a)).

Suppose C is a Hamilton cycle in G. Then $S = N(s) - \{v_2, v_3, u_2, u_3\}$ (i.e. the set of unlabelled neighbours of s in H in Figure 1 (a)) is an independent set of cardinality four and $N(S) = \{v_2, v_3, u_2, u_3, s\}$. The intersection of C with $\langle N[s] \rangle$ is therefore a path with end vertices in $\{v_2, v_3, u_2, u_3\}$. Hence any path cover of H contains at most one path that has one end vertex in $\{u_1, u_2, u_3\}$ and the other in $\{v_1, v_2, v_3\}$. Thus every Hamilton cycle in G has at most one path from Z_1 to Z_2 that passes through the border between them. Therefore, since each node has three borders incident to it, if G' is not hamiltonian, then G is not hamiltonian. □

It follows from Theorems 3.3.1 and 3.3.5 that $\Delta^*_{LH} \in \{7, 8\}$. I think it very unlikely that connected nonhamiltonian LH graphs with maximum degree 7 exist, and speculate that there are only finitely many connected nonhamiltonian LH graphs with maximum degree 8, which would imply that $\Delta^*_{LH} = 8$.

Finally, a note on toughness. Chvátal raised the question of whether maximal planar nonhamiltonian graphs can be 1-tough [24]. This was answered by Nishizeki [24] by exhibiting such a graph of order 19 and maximum degree 15. Soon afterwards, Dillencourt [14] and Tkáč [32] found smaller examples of such graphs (orders 15 and 13 respectively, with maximum degree 9). Tkáč also showed that 13 is the smallest possible order for such graphs. Tkáč's graph can be found in Figure 3.13. It is still unknown whether a connected LH graph with maximum degree 8 can be nonhamiltonian but 1-tough.

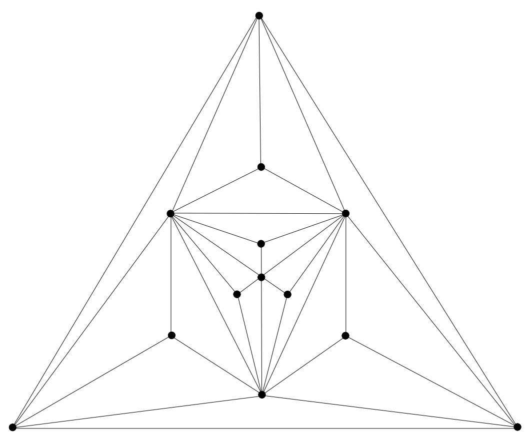

Figure 3.13: A 1-tough maximal planar graph of order 13 with maximum degree 9.

3.4 Traceability of Locally Hamiltonian Graphs

The material in this section has been published in [34].

I begin this section by addressing Question 2: Is 14 the smallest order of a connected nontraceable LH graph?

As mentioned earlier, the graph in Figure 3.1 is a connected nontraceable LH graph of order 14. Thus it remains to prove that every LH graph of order less than 14 is traceable.

From Theorem 3.3.1 it follows that if G is a connected nonhamiltonian LH graph, then $\Delta(G) \geq 7$.

Note that if w is any vertex in an LH graph, then $\langle N[w] \rangle$ contains a wheel with centre w. The following two results concerning wheels will be used extensively throughout the proof of our main result in this section.

Lemma 3.4.1. *Let W be a wheel of order $d+1$, $d \geq 3$ with centre vertex w and rim C denoted by $v_1 \dots v_d v_1$. Then W has a Hamilton path between v_i and v_j, for every pair i, j with $1 \leq i < j \leq d$. Moreover every edge of C lies on some Hamilton path between v_i and v_j except for the edge $v_i v_j$ (when $j = i + 1$).*

Figure 3.14 illustrates the Hamilton paths in Observation 3.4.2 for the cases (b), (c) and (d) .

We define a k-path cover of a graph G to be a set of k disjoint paths that contain all the vertices in G.

Observation 3.4.2. *Suppose a graph G contains a wheel W with centre vertex w and rim C, denoted by $v_1 \dots v_d v_1$. Suppose $G - V(W)$ has a k-path cover $Q_1, \dots, Q_k$. Let a_i, b_i be the end-vertices of Q_i, $i = 1 \dots, k$. (If Q_i is a singleton, then $a_i = b_i$.) Then the following hold.*

(*a*) *If $k = 1$ and a_1 has a neighbour in C, then G is traceable.*

(*b*) *If $k = 2$ and C contains a pair of distinct vertices $\{u_1, u_2\}$ such that $u_i \in N(a_i)$, $i = 1, 2$, then G is traceable.*

(*c*) *Suppose $k = 3$ and C contains two distinct pairs of distinct vertices $\{u_1, v_1\}$ and $\{u_2, u_3\}$ such that $u_i \in N(a_i)$ for $i = 1, 2, 3$ and $v_1 \in N(b_1)$. Then G is traceable if the set $\{u_1, v_1, u_2, u_3\}$ contains two consecutive vertices of C.*

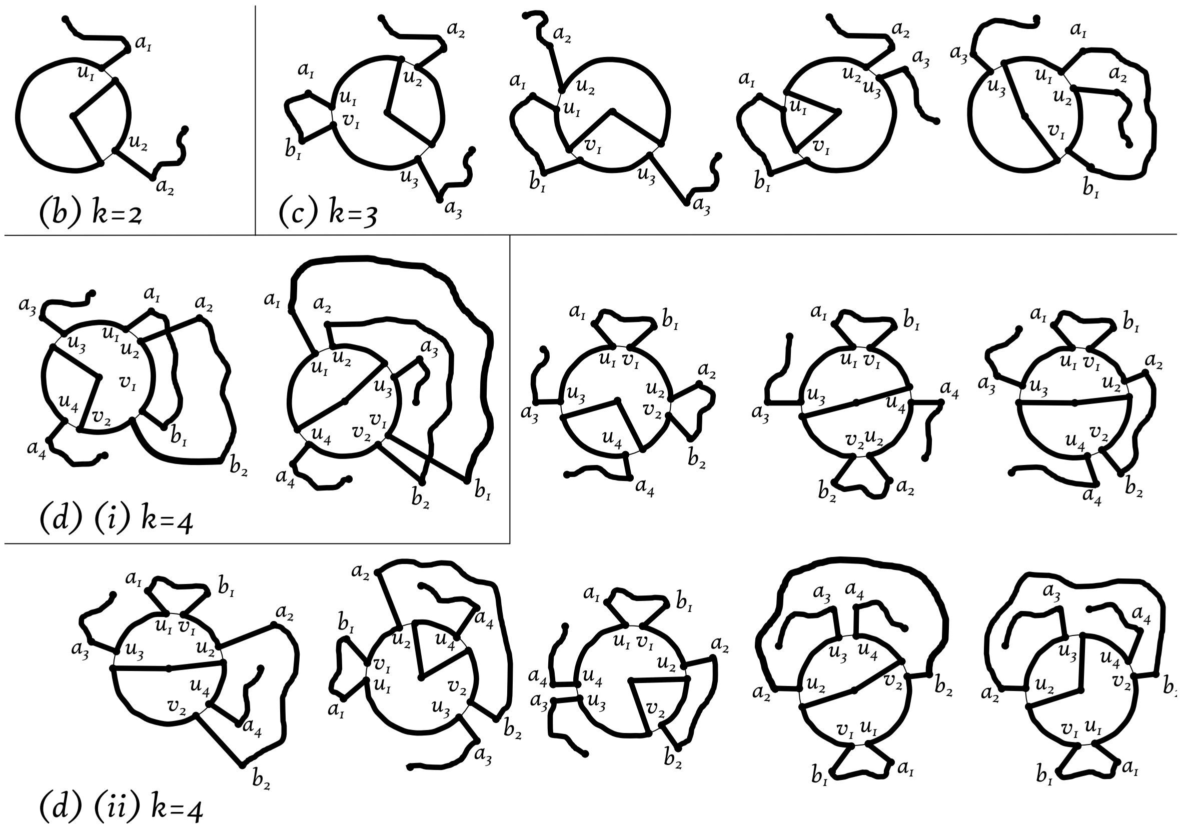

Figure 3.14: The Hamilton paths referred to in Observation 3.4.2.

(*d*) *Suppose* $k = 4$ *and* C *contains three distinct pairs of distinct vertices* $\{u_1, v_1\}$, $\{u_2, v_2, \}$ *and* $\{u_3, u_4\}$ *such that* $u_i \in N(a_i)$ *for* $i = 1, 2, 3, 4$ *and* $v_i \in N(b_i)$ *for* $i = 1, 2$. *Then* G *is traceable if either of the following hold.*

(*i*) *The vertices* u_2 *and* v_2 *are the respective successors of* u_1 *and* v_1 *on* C.

(*ii*) *The vertices* u_1 *and* v_1 *are consecutive vertices of* C *and the set* $\{u_2, v_2, u_3, u_4\}$ *contains a pair of consecutive vertices of* C.

Note that by "distinct pairs of distinct vertices" we mean that the two vertices in a given pair are distinct and any two given pairs have at most one vertex in common.

Lemma 3.4.1 implies that an LH graph of order n with maximum degree $n - 2$ is hamiltonian. Adding a vertex (with any number of edges incident to it) to a hamiltonian graph results in a traceable graph. We thus get the following.

Corollary 3.4.3. *If* G *is a connected nontraceable* LH *graph, then* $\Delta(G) \leq n - 4$.

Lemma 3.4.4. *Suppose G is a connected LH graph. For any $w \in V(G)$, let $C = v_1v_2 \ldots v_dv_1$ be a Hamilton cycle in $\langle N(w) \rangle$ and let $X = G - N(w)$. Let S be the union of any s components of X. Then the following hold.*

(i) If for some $v_i \in N(w)$, v_i has at least one neighbour in each component of S, then $|N_C(v_i) \cap N_C(V(S))| \geq s+1$ and $|N_C(V(S))| \geq s+2$.

(ii) If $s \in \{2, 3\}$, then $|N_C(V(S))| \geq s+2$.

Proof. (i) Since $\langle N_S(v_i) \cup \{w\} \rangle$ has at least $s+1$ components, and since $\langle N(v_i) \rangle$ is hamiltonian, $\langle N(v_i) - \{w\} \rangle$ has a Hamilton path P with initial and terminal vertices on C. Since the maximal subpaths of P that intersect each component of S are preceded and followed by vertices on C, $|N_C(v_i) \cap N_C(V(S))| \geq s+1$, and since $v_i \in N_C(V(S))$, the result follows.

(ii) Suppose $|N_C(V(S))| \leq s+1$. Since G is 3-connected, each component of S has at least 3 neighbours on C, and so, if $s \in \{2, 3\}$, it follows from the pigeonhole principle that there is some vertex v_i on C that has a neighbour in each component of S. The result follows from (i).

□

The following observation will be used extensively in the proof of our main result in this section.

Observation 3.4.5. *If H is a connected graph of order $n \leq 5$, then one of the following holds.*

(a) H is hamiltonian.

(b) H is nonhamiltonian but traceable and H has a Hamilton path Q with end-vertices a, b such that $d(a) \leq 1$, $d(b) \leq 2$ if $n \leq 4$ and $d(a) \leq 2$ if $n = 5$.

(c) H is nontraceable and has a 2-path cover Q_1, Q_2, such that Q_i has an end-vertex a_i of degree 1 for $i = 1, 2$, and all the end-vertices of Q_1 and Q_2 are independent.

(d) $H = K_{1,4}$.

Figure 3.15 shows the connected nontraceable graphs of order $n \leq 5$.

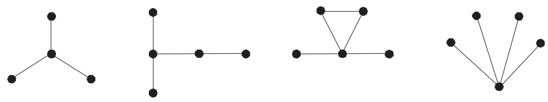

Figure 3.15: The connected nontraceable graphs of order $n \leq 5$.

Theorem 3.4.6. *Suppose G is a connected LH graph of order $n \leq 13$. Then G is traceable.*

Proof. Suppose to the contrary that G is a connected nontraceable LH graph of $n \leq 13$. Let w be a vertex in G of degree $d = \Delta(G)$, let $C = v_1 \ldots v_d v_1$ be a Hamilton cycle in $\langle N(w) \rangle$ and $X = G - N[w]$. By Theorem 3.3.1 and Corollary 3.4.3, $\Delta(G) \in \{7, 8, 9\}$.

Suppose $\Delta(G) = 9$. Then $|V(X)| \leq 3$. If $E(X) \neq \emptyset$, then since G is 3-connected, it follows from Observation 3.4.2(a) and (b) that D is traceable. If $E(X) = \emptyset$, it follows from Lemma 3.4.4(ii), that X has at least two consecutive neighbours on C. Hence, since G is 3-connected, Observation 3.4.2(c) implies that G is traceable. We may therefore assume $\Delta(G) \in \{7, 8\}$.

Now let $Q_1, \ldots, Q_k$ be a minimum path cover of X and let a_i, b_i be the end-vertices of Q_i, $i = 1 \ldots, k$. (If Q has only one vertex, then $a_i = b_i$.) Since $Q_1, \ldots, Q_k$ is a minimum path cover of X, $a_i a_j, b_i b_j, a_i b_j \notin E(G)$ for $i \neq j$.

Claim 1: *If $v_i \in C$, then v_i is adjacent to at most 2 components of X.*

Proof of Claim 1: By Lemma 3.1.5, v_i is adjacent to at most $\frac{\Delta(G)}{2} - 1$ components in X, and since $\Delta(G) \in \{7, 8\}$, we need only consider the case where $\Delta(G) = 8$ and some $v_i \in C$ is adjacent to exactly three components in X. Hence if $k = 3$ then $V(X) = \{a_1, a_2, a_3\}$ or $V(X) = \{a_1, a_2, a_3, b_3\}$, otherwise $k = 4$ and $V(X) = \{a_1, a_2, a_3, a_4\}$. Without loss of generality we may assume $\{a_1, a_2, a_3\} \subset N(v_1)$. Since $\Delta(G) = 8$, it follows from Lemma 3.4.4(i) that v_1 has exactly 4 neighbours on C. Since $\{a_1, a_2, a_3, w\}$ is an independent set in $\langle N(v_1) \rangle$, and since $\langle N(v_1) \rangle$ is Hamiltonian, there exists an a_i and a_j in $N(v_1)$, $a_i \neq a_j$, such that $a_i \in N(v_8)$ and $a_j \in N(v_2)$. But, since G is 3-connected, this contradicts Observation 3.4.2(c) if $k = 3$ and it contradicts Observation 3.4.2(d)(ii) if $k = 4$.

We now consider the k-path cover $Q_1, ..., Q_k$ of X. There are five cases to consider.

Case $k = 1$.

Since G is 3-connected, it follows from Observation 3.4.5(a) and (b) that an end-vertex of Q_1 has a neighbour on C. Hence by Observation 3.4.2, G is traceable.

Case $k = 2$.
Since G is 3-connected, it follows from Observation 3.4.5(a), (b) and (c) that there are two distinct vertices u_1 and u_2 on C such that u_i is adjacent to an end-vertex of Q_i, $i = 1, 2$. Hence, by Observation 3.4.2, G is traceable.

Case $k = 3$.
If X is a star $K_{1,4}$, and x its central vertex, then $\alpha(\langle N(x)\rangle) = 4$, which contradicts Lemma 3.1.5, since in this case $\Delta(G) = 7$. Hence X has either 2 or 3 components and each component of X has at most 4 vertices and at least one component is a singleton. Thus we may assume that $Q_1 = \{a_1\}$ and that a_1 has three distinct neighbours on C. Moreover, by Observation 3.4.5(a), (b), (c) and the fact that G is 3-connected, we may assume that either each of a_2 and a_3 has at least two neighbours on C or a_2 has at least three neighbours on C and a_3 has at least one neighbour on C. If a neighbour of a_3 (or b_3) is the successor or predecessor of a neighbour of a_2 (or b_2) on C, it follows from Observation 3.4.2(c) that G is traceable. Also if two of the neighbours of a_1 are consecutive on C, Observation 3.4.2(c) implies that G is traceable.

It remains to consider the case where no neighbour of a_i is a successor or predecessor of a neighbour of a_j (or b_j) on C for $i \neq j$, and if $a_i = b_i$, a_i has no consecutive neighbours on C.

If $\Delta(G) = 7$ we may therefore assume that $N(a_1) = \{v_1, v_3, v_5\}$ and since $\langle N(a_1)\rangle$ is hamiltonian, $v_1v_3, v_1v_5, v_3v_5 \in E(G)$. Since the set $\{a_2, a_3\}$ has at least four neighbours in $\{v_1, v_3, v_5\}$, at least one of v_i, $i = 1, 3, 5$, is of degree 8, a contradiction.

Hence $\Delta(G) = 8$ and $n(V(X)) \leq 4$ and $V(X) = \{a_1, a_2, a_3\}$ or $V(X) = \{a_1, a_2, a_3, b_3\}$. But now $|N_C(V(X))| = 4$, contradicting Lemma 3.4.4(ii).

Case $k = 4$.
If $n(X) = 4$, $V(X) = \{a_1, a_2, a_3, a_4\}$ and if $n(X) = 5$, $V(X) = \{a_1, a_2, a_3, a_4, b_4\}$. Observe also that since $\delta(G) \geq 3$, there are at least 12 edges between $V(C)$ and $V(X)$. We make the following claims.

Claim 2: *If a_k is an isolated vertex in X, and if $v_i \in N(a_k)$, then $v_{i-1} \notin N(a_k)$ and $v_{i+1} \notin N(a_k)$. If $n(X) = 5$ and $v_i \in N(a_4)$, then $v_{i-1} \notin N(b_4)$ and $v_{i+1} \notin N(b_4)$.*

Proof of Claim 2: First suppose $V(X) = \{a_1, a_2, a_3, a_4\}$.

Suppose to the contrary that $\{v_1, v_2\} \subseteq N(a_1)$. By Claim 1 and since G is 3-connected, there are at least seven edges between the $d-2$ vertices in $C - \{v_1, v_2\}$ and $V(X) - \{a_1\}$. By Observation 3.4.2 (d)(ii), no two consecutive vertices on the path $v_3v_4 \dots v_d$ have neighbours in $V(X) - \{a_1\}$. Hence at most $\lceil \frac{d-2}{2} \rceil$ vertices on the path $v_3v_4 \dots v_d$ are neighbours of $X - a_1$. Since $d \in \{7, 8\}$, no more than three such vertices exist. But then one of these vertices has at least three neighbours in $V(X)$, contradicting Claim 1.

Now suppose $V(X) = \{a_1, a_2, a_3, a_4, b_4\}$. Note that in this case $\Delta(G) = 7$.

If $v_1 \in N(a_4)$ and $v_2 \in N(b_4)$, the argument above is directly applicable. So assume without loss of generality that $\{v_1, v_2\} \subseteq N(a_1)$. If $N(a_2) \cap \{v_1, v_2\} = \emptyset$, then by Observation 3.4.2(d)(ii), $N(a_2) = \{v_3, v_5, v_7\}$. Hence, again by Observation 3.4.2(d)(ii), $N_C(\{a_3, a_4, b_4\}) \subseteq \{v_3, v_5, v_7\}$. But then each of v_3, v_5 and v_7 has neighbours in three components of X, contrary to Claim 1.

If $\{v_1, v_2\} \subset N(a_2)$, then by Claim 1 and Observation 3.4.2(d)(ii), $N(\{a_3, a_4, b_4\}) = \{v_4, v_5, v_6\}$. But since $\delta(G) \geq 3$, this again contradicts Claim 1. Therefore a_2, and by symmetry, a_3, each has exactly one neighbour in $\{v_1, v_2\}$. Hence by Observation 3.4.2(d)(ii) $N(a_2, a_3) = \{v_1, v_2, v_4, v_6\}$. This implies that no vertex in $V(C)$ is adjacent to a_4 or b_4 contradicting the fact that G is 3-connected.

Claim 3: *$\Delta(G) = 8$ and $X = \{a_1, a_2, a_3, a_4\}$.*

Proof of Claim 3: Suppose $\Delta(G) = 7$. By Claim 1 and since each component of X has at least three distinct neighbours in $V(C)$, we may assume without loss of generality that v_1 has neighbours in two components of X. Suppose v_1 is adjacent to a_i and a_j where $i, j \neq 4$. Then by Claim 2, $\{a_i, a_j\} \cap N(\{v_2, v_7\}) = \emptyset$. If $n(X) = 5$ and, say $j = 4$, then Claim 2 implies that $\{a_i, b_4\} \cap N(\{v_2, v_7\}) = \emptyset$. By Lemma 3.4.4(i), v_1 has at least three neighbours in $V(C)$ other than v_2 and v_7, and since v_1 is also adjacent to w, $d(v_1) \geq 8$, a contradiction.

Claim 4: *If $v_i \in N(a_1) \cap N(a_2)$, then there exists a $v_j \neq v_i$ such that $v_j \in N(a_1) \cap N(a_2)$.*

Proof of Claim 4: Suppose $\{a_1, a_2\} = N_X(v_1)$. By Claim 2, $\{v_2, v_8\} \cap \{N(a_1) \cup N(a_2)\} = \emptyset$. By Lemma 3.4.4(i) and since $\Delta(G) = 8$, v_1 has exactly three neighbours other than v_2 and v_8 in $V(C)$. Since $\langle N(v_1) \rangle$ is hamiltonian, one of these three neighbours is adjacent to both a_1 and a_2.

Claim 5: *$d(a_i) = 3$ for all $a_i \in X$.*

Proof of Claim 5: Suppose to the contrary that $d(a_1) > 3$. Then by Claim 2, $d(a_1) = 4$ and we may assume without loss of generality that $N(a_1) = \{v_1, v_3, v_5, v_7\}$. By Observation 3.4.2(d)(i) at most one of $\{v_2, v_4, v_6, v_8\}$ is in $N(a_i)$, $a_i \neq a_1$. Since $\delta(G) \geq 3$ this implies that each $a_i \neq a_1$ is adjacent to at least two vertices in $N(a_1)$, contradicting Claim 1.

We can now proceed with the main proof of the theorem.

By Claim 5 there are 12 edges between $V(C)$ and $V(X)$. Hence by Claim 1 we may assume without loss of generality that $N_X(v_1) = \{a_1, a_2\}$. By Claims 2 and 5 we may also assume that either $N(a_1) = \{v_1, v_3, v_5\}$ or $N(a_1) = \{v_1, v_3, v_6\}$. By Claim 4, $|N(a_1) \cap N(a_2)| \geq 2$. Hence, by Claim 1 we may assume that for at least one of a_3 and a_4, say a_4 has no neighbour in $N(a_1)$. Furthermore, by Observation 3.4.2(d)(i), no two neighbours of a_4 are both successors (or both predecessors) of neighbours of a_1 on C. Also, by Claim 2, no two neighbours of a_4 are consecutive vertices on C. But then $d(a_4) < 3$, a contradiction.

Case $k = 5$.

In this case $X = \{a_1, a_2, a_3, a_4, a_5\}$ and $\Delta(G) = 7$. Since $d(a_i) \geq 3$ there are at least 15 edges between $V(C)$ and $V(X)$. But then some v_i on C is adjacent to at least three components in X contradicting Claim 1. □

I conclude that 14 is indeed the smallest order of a connected, nontraceable *LH* graph.

We now turn our attention to constructing nontraceable *LH* graphs with various properties. Triangle identification will be used repeatedly. Note that the Goodey graph (the connected, nontraceable *LH* graph of order 14 in Figure 3.1) has maximum degree 8. Figure 3.16 shows a different depiction of the Goodey graph G. Note that $d(v_1) = 8$ and $\langle G - N(v_1) \rangle \cong K_{1,4}$.

Theorem 3.4.7. *There exists a connected planar nontraceable LH graph of order n with $\Delta(G) \leq 10$ for every $n \geq 14$.*

Proof. First note that the nontraceable *LH* graph of order 14 in Figure 3.16 is planar. This is the same graph as shown in Figure 3.1 redrawn in a more convenient representation. Also note the three vertices of the *LH* graphs constructed in

Observation 3.3.2 that border the outer plane are suitable for use in triangle identification. Label these three vertices u, v and w having degrees 3,4 and 5, respectively. By identifying u with v_5 in Figure 3.16, v with v_2, and w with u_5, we get a planar nontraceable LH graph G with maximum degree of 10. If we start with an LH graph H from Observation 3.3.2 of order $k, k \geq 4$, then $n(G) = 11 + k$. □

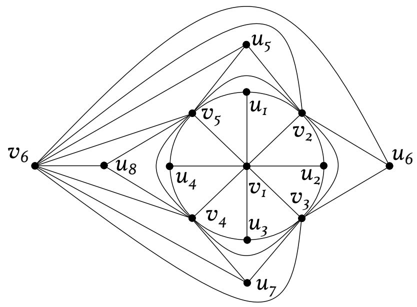

Figure 3.16: The order 14 nontraceable LH graph shown in Section 1 in a different representation. Note that $d(v_1) = 8$ and $\langle G - N(v_1) \rangle \cong K_{1,4}$.

Theorem 3.4.8. *For any integer $k \geq 3$ there exists a nontraceable LH graph G with $\delta(G) = k$.*

Proof. To construct such a graph we start with the order 14 nontraceable LH graph H shown in Figure 3.16. Since complete graphs of order greater than 3 are LH, we can construct the graph G by combining multiple copies of K_{k+1} with G by means of triangle identification in such a way that each vertex of H is used at least once in a triangle identification procedure. Since a triangle can be used at most once in triangle identification (Remark 3.2.6), we must use a new triangle for each step.

Specifically, the triangles formed by edges between the vertices in the following sets in $V(H)$ can be used: $\{v_1, u_1, v_2\}$, $\{v_1, u_2, v_3\}$, $\{v_1, u_3, v_4\}$, $\{v_1, u_4, v_5\}$, $\{v_2, v_3, u_6\}$, $\{v_3, v_4, u_7\}$, $\{v_5, v_6, u_8\}$, and $\{v_5, v_2, u_5\}$. This results in the graph in Figure 3.17 (in this case K_5 was used for the triangle identification, so the minimum degree is 4). □

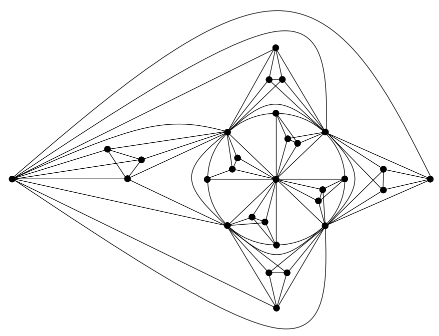

Figure 3.17: A nontraceable LH graph with minimum degree 4.

3.5 Regular connected nonhamiltonian LH graphs

The material in this section has been submitted for publication in [35].

Regular connected LH graphs have not yet received much attention in the literature, except in terms of 6-regular triangulations of the torus [6, 31]. The hamiltonicity of such graphs is readily implied by Theorem 3.3.1.

Questions 3 and 4 by Pareek and Skupień [27] regarding regular LH graphs mentioned in Section 1 are both answered by the following theorem.

Theorem 3.5.1. *For every $r \geq 11$, there exists a nonhamiltonian LH r-regular graph with connectivity 3.*

Proof. To construct an 11-regular connected, nonhamiltonian LH graph R_{11} we start with the Goldner-Harary graph $G11$ shown in Figure 3.18 with the vertices labeled as shown. We then use triangle identification to combine $G11$ with other LH graphs that have the required degree sequences so that the resulting graph is 11-regular. These graphs are shown as graphs $H11A$ and $H11B$ in Figure 3.19 and were constructed by starting with the triangle $\langle\{w_1, w_2, w_3\}\rangle$ and then adding edges linking it to a K_{12} or K_{13} as shown. To limit the degrees of the vertices making up the K_{12} or K_{13} subgraphs to 11, edges were removed between some of these vertices, as indicated in Figure 3.19. It is routine to confirm that these graphs are LH and that the triangle $\langle\{w_1, w_2, w_3\}\rangle$ in each of these graphs is suitable for use in triangle identification. In particular we create the graph R_{11} by combining $G11$ with five copies of $H11A$ and one copy of $H11B$, each time identifying the vertices w_1, w_2, w_3 with appropriate vertices in $G11$. Note that in each step the degrees of the vertices in $G11$ that are identified with w_1, w_2, w_3 of $H11A$ increase by 1, 2, 8, respectively,

Vertices in $G11$	Second graph
v_4, v_2, v_6	$H11A$
v_5, v_1, v_8	$H11A$
v_3, v_4, v_9	$H11A$
v_1, v_4, v_{10}	$H11A$
v_2, v_5, v_{11}	$H11A$
v_5, v_3, v_7	$H11B$

Table 3.2: Details of 11-regular construction for Theorem 3.5.1.

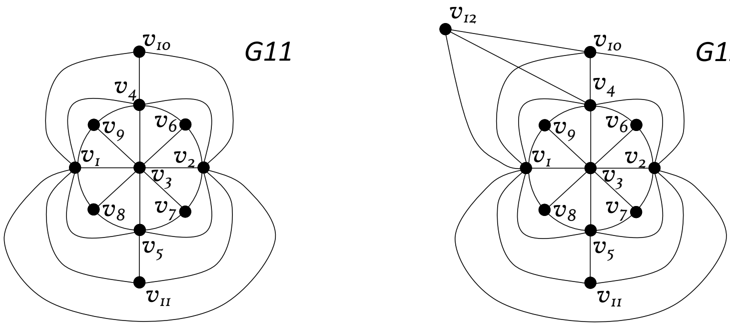

Figure 3.18: The graphs $G11$ and $G12$ used in to construct regular nonhamiltonian LH graphs.

while the degrees of those that are identified with w_1, w_2, w_3 of $H11B$ increase by $2, 2, 8$, respectively. Table 3.2 provides the details of the construction. The first column indicates the first, second and third vertices of the triangle in $G11$ that are identified, respectively, with the vertices w_1, w_2, w_3 of the graph in the second column.

The resulting graph is 11-regular and by Lemma 3.2.2 is connected, nonhamiltonian, and LH. Since it was obtained by means of triangle identification, it has connectivity 3. This technique can easily be extended to create r-regular, connected, nonhamiltonian LH graphs for odd values of r greater than 11. Due to problems with vertex degree parity, the technique does not work for even values of r when starting with graph $G11$. For even values of r greater than or equal to 12 we can use graph $G12$ in Figure 3.18. To create a 12-regular, connected, nonhamiltonian LH graph R_{12} we combine $G12$ with two copies of $H12A$, three copies of $H12B$ and one copy of $H12C$. The details are given in Figure 3.19 and Table 3.3. □

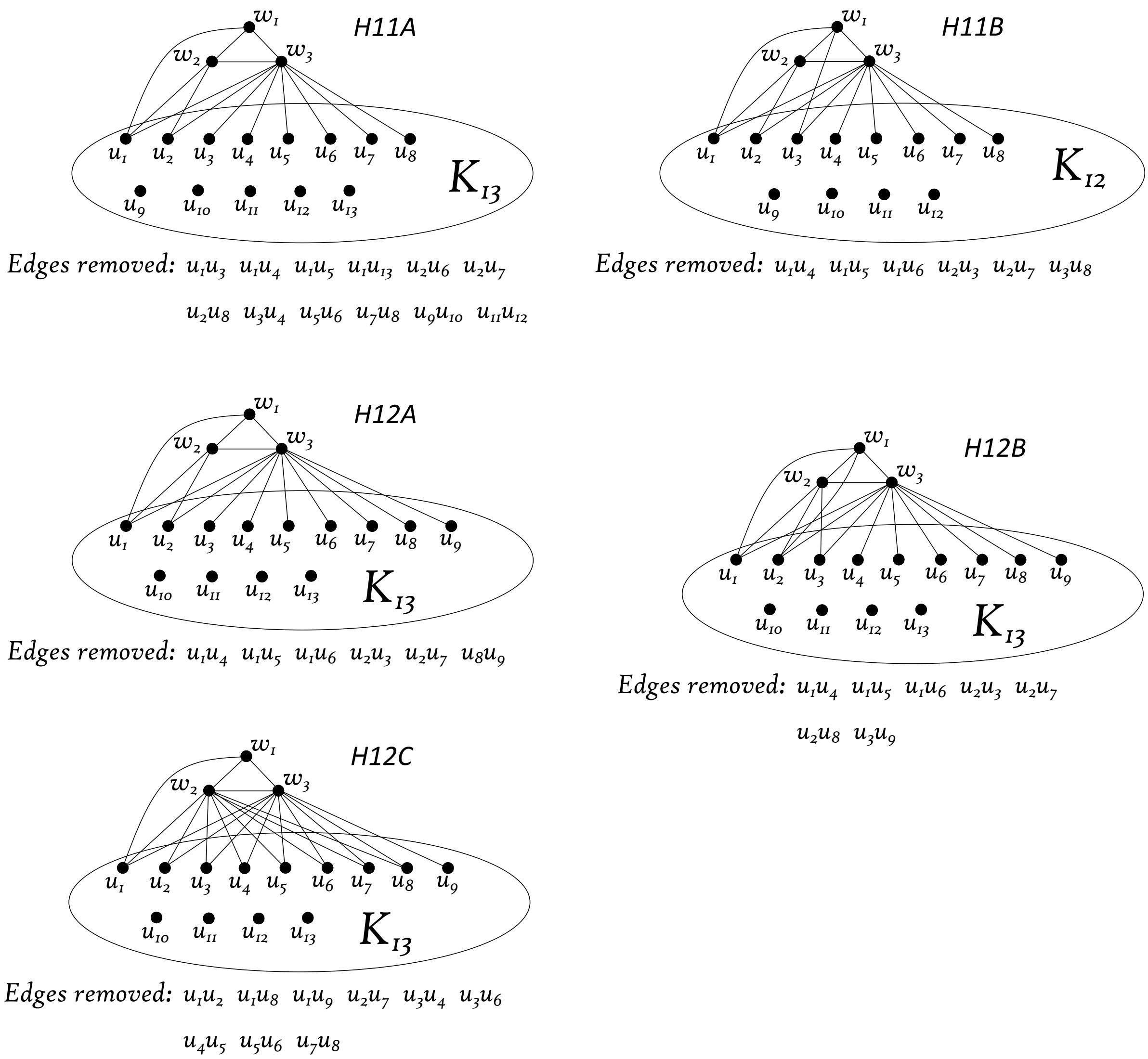

Figure 3.19: The graphs used to construct regular nonhamiltonian LH graphs in combination with $G11$ and $G12$.

Vertices in $G12$	Name of second graph
v_3, v_5, v_7	$H12A$
v_2, v_5, v_{11}	$H12A$
v_5, v_3, v_8	$H12B$
v_4, v_2, v_6	$H12B$
v_4, v_1, v_9	$H12B$
v_4, v_{10}, v_{12}	$H12C$

Table 3.3: Details of 12-regular construction for Theorem 3.5.1.

3.6 Longest paths in LH graphs

The material in this section has been submitted for publication in [35].

The title of this section comes from a paper by Entringer and MacKendrick [16]. For $n \geq 4$, they define $f(n)$ to be the largest integer such that every connected LH graph on n vertices contains a path of length $f(n)$. They established the following upper bound for $f(n)$.

Theorem 3.6.1. [16] $f(n) \leq 24\sqrt{n/3} + 4$ *for* $n \geq 4$.

Although Entringer and MacKendrick did not explicitly state it, the following corollary is an obvious implication of Theorem 3.6.1.

Corollary 3.6.2. $\lim_{n\to\infty} \frac{f(n)}{n} = 0$.

The LH graphs constructed by Entringer and MacKendrick to provide the bound in Theorem 6.1 are nonplanar and there is no restriction on their maximum degree. However, it is possible to prove a result equivalent to Corollary 3.6.2 for planar graphs with bounded maximum degree. We define $p(n, \Delta)$ to be the largest integer such that every connected planar LH graph of order n with maximum degree Δ contains a path of length $p(n, \Delta)$. I now prove the following result, which is stronger than Corollary 3.6.2.

Theorem 3.6.3. $\lim_{n\to\infty} \frac{p(n,\Delta)}{n} = 0$ *for every* $\Delta \geq 11$.

Proof. Consider the order 23 graph G_0 shown in Figure 3.20. This graph is constructed from the Goldner-Harary graph (G11A in Figure 3.5 and also the first graph in Figure 3.21) by adding 12 vertices using repeated triangle identification with copies of K_4. Clearly $\Delta(G_0) = 11$ and by Lemma 3.2.2 G_0 is LH, planar and nonhamiltonian. Let the K_3 subgraphs of G_0 that are encircled in Figure 3.20 be labeled $H_1, H_2, \ldots, H_6$ as shown. G_0 is traceable, but it should be noted that there is no Hamilton path that starts in H_i and ends in H_i, $i \in \{1, 2, 3, 4, 5, 6\}$. Now let the graphs $G_{0,1}, G_{0,2}, \ldots, G_{0,6}$ be six copies of G_0, each with the K_3 subgraphs labeled in the same way as in G_0. Use triangle identification to combine G_0 with $G_{0,i}$ by identifying H_i in G_0 with H_i in $G_{0,i}$, $i = 1, 2, 3, 4, 5, 6$, to create the graph G_1 (This is possible, since each H_i contains a vertex that is of degree 3 in G_0 and in $G_{0,i}$). Also note that $\Delta(G_1) = 11$ and that G_1 is planar. Since each $G_{0,i}$ contains a vertex cutset of order 5, it follows that a longest path in G_1 omits one H_j subgraph in four of the subgraphs represented by $G_{0,i}$ so that the longest path in G_1 has length $23+2\times20+4\times17 = 131$, while $n(G_1) = 23+6\times20 = 143$. One can now repeat the procedure by combining G_1 with 6×5 copies of G_0 in the same way to create the graph G_2. A longest path in G_2 contains $23+2\times20+4\times17+2\times20+6\times4\times17 = 579$ vertices, while $n(G_2) = 23+6\times20+6\times5\times20 = 743$. This process can be continued indefinitely. By Lemma 3.2.2 (b) the graph G_k is planar and $\Delta(G_k) = 11$, while the longest path in G_k contains $p_k = 23 + 2 \times 20 + 4 \times 17 + \sum_{i=2}^{k}(2 \times 20 + 6 \times 4^{i-1} \times 17)$ vertices, while $n(G_k) = 23 + \sum_{i=1}^{k} 6 \times 5^{i-1} \times 20$. It is then easy to show that $\lim_{k\to\infty} \frac{p_k}{n(G_k)} = 0$ and the result follows for $\Delta = 11$. The result can easily be extended to greater values for the maximum degree by combining the graph G_k with a planar graph with the required maximum degree by triangle identification with one of the outer triangle subgraphs. □

Note that Entringer and MacKendrick's limit only implies the existence of connected nontraceable LH graphs of order greater than or equal to 200. However, Theorem 3.4.6 states that the smallest connected nontraceable LH graph has order 14, so there is much room for improvement for low values of n. Our next theorem provides an upper limit for $f(n)$ that is smaller than the one given by Entringer and MacKendrick for $n \leq 427$ and implies that $f(n) < n$ for every $n \geq 15$.

Theorem 3.6.4. $f(n) \leq \lceil (2/3)n \rceil + 4$.

Proof. Consider the graph G_0 shown in Figure 3.21. This is the Goldner-Harary graph shown in Figure 3.5 (a), redrawn to emphasize the fact that the six vertices

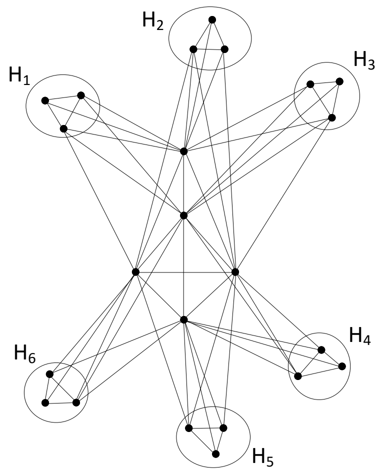

Figure 3.20: The graph G_0 used in Theorem 3.6.3.

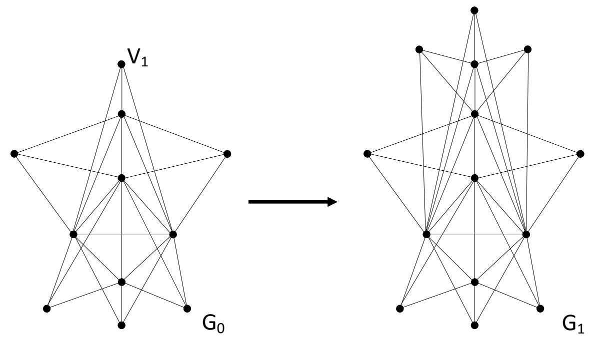

Figure 3.21: The graphs G_0 and G_1 used in Theorem 3.6.4.

of degree 3 are connected to each other by a cutset of 5 vertices. Now choose any vertex of degree 3, call it v_0, and using Lemma 3.2.2 use triangle identification to combine G_0 with three copies of K_4, each time using v_0 and two of its neighbours, to create the graph G_1. G_1 now has a vertex cutset of order six (v_1 is now also in the cutset), the removal of which results in eight components. In general, the graph G_{i-1} can be combined with three copies of K_4 using any vertex of degree 3 in $V(G_{i-1})$, call it v_{i-1}, to create the graph G_i. By Lemma 3.2.2 (a) and (b), G_i is LH and planar. Also, G_i has a vertex cutset of order $5+i$, the removal of which results in a graph consisting of $6+2i$ isolated vertices. It follows that a longest path in G_i has no more than $2(5+i)+1$ vertices, and that $n(G_i) = 11+3i$. Let $q(n)$ be the number of vertices in a longest path in a graph on n vertices constructed in this way (where the last vertex v_i to be used in triangle identification may have been used once, twice, or three times). Then $q(n) \leq \lceil (2/3)n \rceil + 4$. □

Chapter 4

Nested Locally Hamiltonian Graphs

4.1 Introduction

We call a graph G *locally locally connected* (written LLC or L^2C) if $\langle N(v) \rangle$ is an LC graph for every $v \in V(G)$. We extend this concept in a natural way to L^kC graphs for $k = 0, 1, 2, \ldots$ (where L^0C simply means *connected*). L^kH graphs are defined analogously. (Formal definitions for these concepts are provided in Section 4.3.)

For each $k \geq 0$, the class of L^kH graphs contains an interesting subclass, namely the class of SC $(k+2)$-trees. (This is shown in Section 4.3.) Recall that the class of SC 2-trees are exactly the maximal outerplanar graphs, and are therefore L^0H, while the SC 3-trees are exactly the chordal maximal planar graphs, and are therefore LH - See Corollaries 2.1.5 and 3.1.7.

Our interest in L^kH and L^kC graphs was sparked by Theorem 1.2.7 by Oberly and Sumner and their conjectured extensions of the theorem (Conjectures 1.2.8 and 1.2.9).

An LH graph is locally 2-connected, so the following conjecture is weaker than the case $k = 2$ of Conjecture 1.2.8.

Conjecture 4.1.1. *If G is a connected $K_{1,4}$-free LH graph, then G is hamiltonian.*

Let G be a connected, nonhamiltonian LH graph of order n. By Lemma 3.1.2, $\Delta(G) \leq n - 3$. If G contains an induced $K_{1,4}$ with v as its central vertex, then the fact that $\langle N(v)\rangle$ is hamiltonian implies that $d(v) \geq 8$, so that $n \geq 11$. Thus, if Conjecture 4.1.1 is true, it would imply that every connected, nonhamiltonian, locally hamiltonian graph has maximum degree at least 8 and order at least 11. Pareek and Skupień [27] proved that the minimum order of nonhamiltonian, connected *LH* graphs is indeed 11. It is shown in Section 3.3 that there are four nonhamiltonian, connected *LH* graphs of order 11 and they all have maximum degree 8. Pareek [26] claimed that every nonhamiltonian connected *LH* graph has maximum degree at least 8, but there are flaws in his "proof" that I have not been able to rectify, as discussed in Section 3.3. If Conjecture 4.1.1 is true, it would immediately prove Pareek's (as yet unproved) claim.

I shall show that if G is an L^kH graph that is L^mC for $m = 0, 1, \ldots, k-1$, then G is locally $(k+1)$-connected. This motivated us to consider the following conjecture, which extends Conjecture 4.1.1 and is weaker than Conjecture 1.2.8.

Conjecture 4.1.2. *If G is an L^kH graph that is L^mC for $m = 0, 1, \ldots, k-1$ and G contains no induced $K_{1,k+3}$, then G is hamiltonian.*

Remark 4.1.3. *In order to exclude trivial cases in our study of the hamiltonicity of L^kH graphs, I added the requirement that they be L^mC for $k = 0, 1, \ldots, k-1$. This is analogous to limiting investigations on the hamiltonicity of LH graphs to the connected case. The graph consisting of two copies of K_5 sharing a common vertex is an example of an LLH graph that is connected but not LC and is trivially nonhamiltonian.*

Graphs satisfying the hypothesis of Conjecture 4.1.2 have a rich and regular structure. In Section 4.2 I study *LLH* graphs that are connected and *LC* and I develop means of constructing and manipulating such graphs to obtain ones with prescribed properties. I show that the minimum order of a nonhamiltonian *LLH* graph that is connected and *LC* is 13. Note that if Conjecture 4.1.2 is true, it would imply that a nonhamiltonian graph that is *LH*, L^kH and L^mC for $m = 0, 1, 2, \ldots, k-1$, has maximum degree at least $6+2k$ and hence order at least $9+2k$ (by Lemma 3.1.2). In Section 4.3, for each $k \geq 1$, I construct nonhamiltonian graphs of order $9+2k$ that are L^mH for $m = 1, 2, \ldots, k$, as well as nonhamiltonian L^kH graphs of order $9+2k$ that are not L^mH for $m = 1, 2, \ldots, k-1$. It is worth noting that these graphs are locally $(k+1)$-connected and all contain an induced $K_{1,k+3}$ as Conjecture 4.1.2 requires, but as will be shown, do not contain an induced $K_{1,k+4}$. This implies that if the Oberly-Sumner conjecture is true, it would be best possible in a very strong sense.

I also construct a sequence of L^kH graphs that are L^mC, $m = 0, 1, \ldots, k-1$ such that the detour order becomes a vanishing fraction of the order of the graph.

Finally, I investigate the NP-completeness of the HCP for L^kH graphs that are L^mC for $m = 0, 1, \ldots, k-1$ and for graphs that are L^mH for $m = 1, 2, \ldots, k$.

4.2 Locally locally hamiltonian graphs

Definition 4.2.1. *A graph G is locally locally hamiltonian (LLH or L^2H) if $\langle N(v) \rangle$ is locally hamiltonian for every $v \in V(G)$.*

The following is an alternative formulation of the above definition and is often more convenient.

Definition 4.2.2. *A graph G is locally locally hamiltonian (LLH or L^2H) if $\langle N(v) \cap N(u) \rangle$ is a hamiltonian graph for every pair of adjacent vertices $u, v \in V(G)$.*

Since $\langle N_{N(v)}(u) \rangle = \langle N(v) \cap N(u) \rangle$, it is clear that these two definitions are equivalent.

Note that a hamiltonian graph has order at least three, since it contains a Hamilton cycle.

Lemma 4.2.3. *Let G be a connected, LLH graph that is also LC. Then G is 4-connected (and hence $\delta(G) \geq 4$).*

Proof. Since connected LH graphs are 3-connected with minimum degree at least 3, it follows that LC, LLH graphs are locally 3-connected and are therefore 4-connected by Theorem 3.1.4, and hence have $\delta \geq 4$. $\square$

In Section 3.2 I developed the concept of triangle identification to combine locally hamiltonian graphs. I now show that a similar technique can be used to combine LLH graphs. We refer to it as K_4-identification.

Construction 4.2.4. *(K_4-identification) For $i = 1, 2$, let G_i be an LLH graph that contains a 4-clique X_i such that for each pair of vertices $x_j, x_k \in V(X_i)$, there is a Hamilton cycle in $\langle N(x_j) \cap N(x_k)\rangle$ that contains the edge $X_i - \{x_j, x_k\}$. Now suppose $V(X_1) = \{v_1, v_2, v_3, v_4\}$, and $V(X_2) = \{u_1, u_2, u_3, u_4\}$. Create a larger graph G by identifying the vertices v_j and u_j, $j = 1, 2, 3, 4$ to a single vertex w_j, while retaining all the edges present in the original two graphs (see Figure 4.1). We say that G is obtained from G_1 and G_2 by identifying suitable K_4's.*

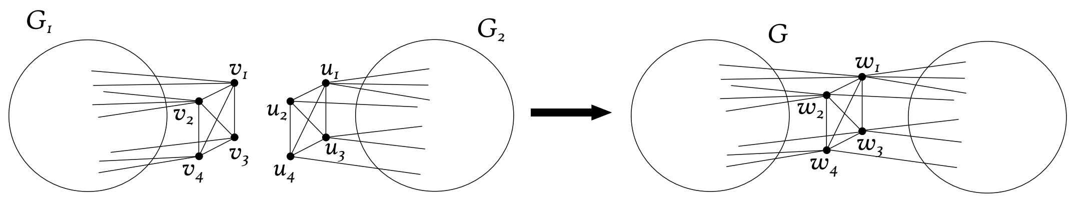

Figure 4.1: The K_4-identification procedure.

The following theorem is a special case of the more general Theorem 4.3.11 presented and proved in Section 4.3.

Theorem 4.2.5. *If two LC, LLH graphs G_1 and G_2 are combined using K_4-identification to form a larger graph G, then G is also LC and LLH.*

Our next result follows immediately from Definition 4.2.1 and Theorem 3.1.3.

Lemma 4.2.6. *If G is an LC, LLH graph that is not LH, then $\Delta(G) \geq 11$.*

Proof. G has a vertex v such that $\langle N(v)\rangle$ is LH but not hamiltonian. Hence $d(v) \geq 11$. □

Theorem 4.2.7. *Let G be a connected nonhamiltonian LC, LLH graph of minimum order. Then $n(G) = 13$.*

Proof. Suppose to the contrary that $n(G) < 13$. If G is not LH, then by Lemma 4.2.6 there exists a $v \in V(G)$ such that $d(v) \geq 11$. Then by Theorem 3.4.6, if $n(G) = 12$, with $\Delta(G) = 11$, $\langle N(v)\rangle$ is traceable and hence G is hamiltonian. We can therefore assume G is also LH.

Let $w \in V(G)$ be a vertex of maximum degree, and let $C = v_0v_1 \ldots v_{\Delta-1}v_0$ be a Hamilton cycle of $\langle N(w)\rangle$. Let $X = \langle V(G) - N[w]\rangle$ with $V(X) = \{x_1, x_2, \ldots, x_r\}$. Note that $\delta(G) \geq 4$ by Lemma 4.2.3. This leads to the following claims.

Claims If the vertices of X form an independent set, and G is nonhamiltonian, then the following hold (indices of v taken modulo $\Delta(G)$).

1. If $\Delta(G) = n-3$, then $\{v_i, v_{i+1}\} \not\subset N(x_j)$.

2. If $\Delta(G) = n-3$, then it is not the case that $v_i \in N(x_j)$ and $v_{i+1} \in N(x_k)$, $j \neq k$.

3. If $\Delta(G) = n-4$, then it is not the case that $\{v_i, v_{i+1}\} \subset N(x_k)$ and $\{v_j, v_{j+1}\} \subset N(x_m)$, where $i \neq j$, and $k \neq m$.

The Hamilton cycles that can be found if these conditions are not met are shown in Figure 4.2.

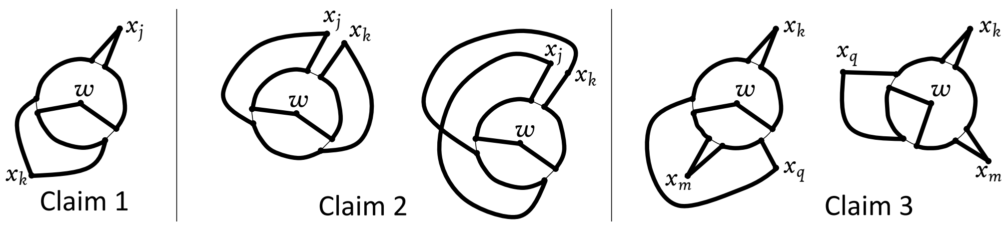

Figure 4.2: The Hamilton cycles that prove the claims in Theorem 4.2.7.

Since G is LH, $n(G) \geq 11$ by Theorem 3.1.3, $\Delta(G) \leq n-3$ by Lemma 3.1.2, and that $\Delta(G) \geq 7$ by Theorem 3.3.1.

Case 1: $n(G) = 11$ and $\Delta(G) = 7$.

$|V(X)| = 3$, so if $comp(X) = 1$, X is traceable and G is obviously hamiltonian. If $comp(X) = 2$, let the components of X be the edge x_1x_2 and the vertex x_3. Because $|N(x_3) \cap N(w)| \geq 4$, $\{v_i, v_{i+1}\} \subset N(x_3)$ for some $i \in \{0, 1, \ldots, 6\}$. Since $|N(x_3) \cap N(w)| \geq 4$, x_3 has two consecutive neighbours on C, and hence G has a Hamilton cycle similar to the one in Claim 1 (with $x_j = x_3$ and the edge x_1x_2 in the place of x_k). Therefore $comp(X) = 3$. Because $N(x_i) \cap N(w) \geq 4$, $i = 1, 2, 3$, each vertex x_i has two successive neighbours in $N(w)$. By Claim 3 we have $\{v_j, v_{j+1}\} \subset N(x_1) \cap N(x_2) \cap N(x_3)$ for some $j \in \{0, 1, \ldots, 6\}$. But then $\{w, x_1, x_2, x_3\}$ is an independent set in $N(v_j)$, so that $d(v_j) \geq 8$.

Case 2: $n(G) = 11$ and $\Delta(G) = 8$.

$|V(X)| = 2$, so if $comp(X) = 1$, X is traceable and G is obviously hamiltonian. If $comp(X) = 2$ then by Claim 1 we have without loss of generality that $N(x_1) =$

$\{v_1, v_3, v_5, v_7\}$ and then by Claim 2 it follows that $N(x_2) = N(x_1)$. Since G is LLH, $\langle N(x_1)\rangle$ is LH and since $d(x_1) = 4$, we get $\langle\{v_1, v_3, v_5, v_7\}\rangle \cong K_4$, so that $d(v_i) = 8$, $i = 1, 3, 5, 7$. Since $\Delta(G) = 8$, v_2 is not adjacent to either of v_5, v_7, otherwise that vertex would have degree greater than 8. If $v_2 \sim v_0$, then $v_1x_1v_7v_6v_5x_2v_3v_4wv_2v_0v_1$ is a Hamilton cycle in G. Hence $|N(v_1)\cap N(v_2)| = 2$ contradicting that $\langle N(v_1)\cap N(v_2)\rangle$ is hamiltonian.

Case 3: $n(G) = 12$.

From Theorem 3.3.4 we know that if $n(G) = 12$ and G is LH and nonhamiltonian, then $\Delta(G) = 9$. Again, $|V(X)| = 2$ and if $comp(X) = 1$, X is traceable and G is clearly hamiltonian, so we can assume $comp(X) = 2$. By Claim 1, we can say without loss of generality that $N(x_1) = \{v_1, v_3, v_5, v_7\}$ and it follows by Claim 2 that $N(x_2) = N(x_1)$. Since G is LLH, $\langle N(x_1)\rangle$ is LH and since $d(x_1) = 4$, we get $\langle\{v_1, v_3, v_5, v_7\}\rangle \cong K_4$. With the exception of v_8v_0 there are no edges in G between vertices in $\{v_2, v_4, v_6, v_8, v_0\}$ if G is nonhamiltonian, as will now be shown.

If $v_2v_4 \in E(G)$, then $v_1x_1v_3v_4v_2wv_6v_5x_2v_7v_8v_0v_1$ is a Hamilton cycle in G.
If $v_2v_6 \in E(G)$, then $v_1x_1v_3v_4v_5x_2v_7v_6v_2wv_8v_0v_1$ is a Hamilton cycle in G.
If $v_2v_8 \in E(G)$, then $v_1x_1v_3v_4v_5x_2v_7v_6wv_2v_8v_0v_1$ is a Hamilton cycle in G.
If $v_2v_0 \in E(G)$, then $v_1x_1v_3v_4v_5x_2v_7v_6wv_8v_0v_2v_1$ is a Hamilton cycle in G.
If $v_4v_6 \in E(G)$, then $v_1v_2v_3x_2v_5v_4v_6wv_0v_8v_7x_1v_1$ is a Hamilton cycle in G.
If $v_4v_8 \in E(G)$, then $v_1x_1v_7v_6v_5x_2v_3v_2wv_4v_8v_0v_1$ is a Hamilton cycle in G.
If $v_4v_0 \in E(G)$, then $v_1v_2v_3x_2v_5v_6wv_4v_0v_8v_7x_1v_1$ is a Hamilton cycle in G.
if $v_6v_8 \in E(G)$, then $v_1x_1v_7x_2v_5v_4v_3v_2wv_6v_8v_0v_1$ is a Hamilton cycle in G.
If $v_6v_0 \in E(G)$, then $v_1v_2wv_6v_0v_8v_7x_2v_5v_4v_3x_1v_1$ is a Hamilton cycle in G.
Since $\delta(G) \geq 4$, it follows that each of v_2, v_4, v_6, v_8, v_0 has an additional neighbour in the set $\{v_1, v_3, v_5, v_7\}$. From the pigeonhole principle it follows that at least one of v_1, v_3, v_5, v_7 has degree at least 10.

It follows that $n(G) \geq 13$.

To see that $n(G) = 13$, note that the graphs in Figure 4.3 (a) and Figure 4.7 are examples of nonhamiltonian LLH graphs of order 13. □

Since we know that the smallest connected nontraceable LH graph has order 14 (Theorem 3.4.6), the next result is somewhat surprising.

Theorem 4.2.8. *Let G be a connected nontraceable LC, LLH graph of minimum order. Then $n(G) = 14$ and if G is not LH, then G has a two-path cover.*

Proof. First note that the graph in Figure 4.3 (b) is a nontraceable, connected LC, LLH graph of order 14. We already know that a connected nontraceable LH graph has order at least 14, so we can assume G is not LH. Then there is a vertex $v \in V(G)$ such that $\langle N(v) \rangle$ is LH but not hamiltonian. It follows that $d(v) \geq 11$. Since all LH graphs of order less than 14 are traceable, $\langle N[v] \rangle$ is hamiltonian, and therefore if $n(G) = 13$, G is traceable, and if $n(G) = 14$, G has a two-path cover. □

Note that the graphs in Figure 4.3 are LLH, but not LH. It is therefore not surprising that $\langle N(w) \rangle$, where w is the vertex shown in Figure 4.3, is the Goldner-Harary graph, which is the smallest connected nonhamiltonian LH graph [27]. A method to construct a connected nonhamiltonian LLH graph of order 13 that is also LH can be found as a special case of the graphs constructed in the proof of Theorem 4.3.24.

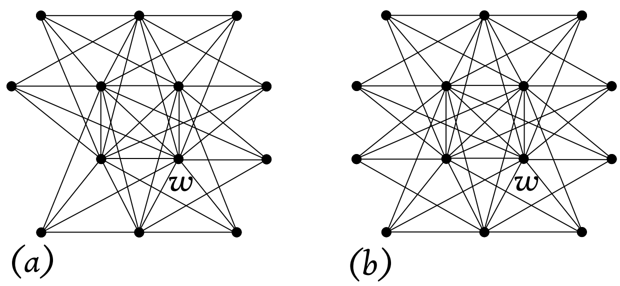

Figure 4.3: (a) nonhamiltonian and (b) nontraceable LLH graphs of orders 13 and 14, respectively.

If G is any nonhamiltonian LH graph, then $\Delta(G) \leq n - 3$ (Lemma 3.1.2), and if G is a nontraceable LH graph, then $\Delta(G) \leq n - 4$ (Corollary 3.4.3).

However, if G is a connected nonhamiltonian LC, LLH graph, then $\Delta(G)$ can be as large as $n - 1$.

The graph in Figure 4.4 is an example of a nonhamiltonian LC, LLH graph of order 15 for which the maximum degree is 14. To see that 15 is the smallest order for which this is possible, note that if G is LLH with $\Delta(G) = n - 1$, there exists a vertex $v \in V(G)$ such that $d(v) = n - 1$ and $\langle N(v) \rangle$ is LH and nontraceable, otherwise G is hamiltonian. Therefore $|N(v)| \geq 14$ and $n(G) \geq 15$.

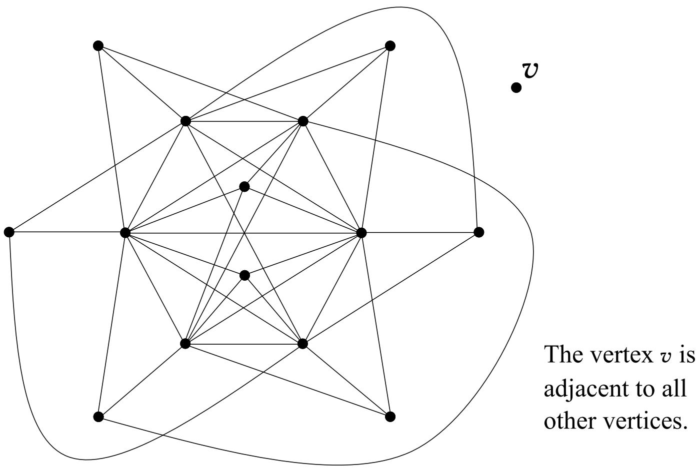

Figure 4.4: A nonhamiltonian LLH graph of order 15 with maximum degree 14.

The following theorem is a special case of Lemma 4.3.14 that is proved in Section 4.3.

Theorem 4.2.9. *Let G_0 be a connected LC, LLH graph that contains a vertex v_1 such that $d(v_1) = 4$. Then $\langle N(v_1) \rangle = K_4$ and v can be used four times in K_4-identification, once in combination with each of the four distinct subsets of three of its neighbours. However, no 4-clique may be used more than once.*

Theorem 4.2.9 can be used to construct nonhamiltonian and nontraceable LC, LLH graphs, such as the two in Figure 4.3. These graphs were constructed by combining two copies of K_5 and then repeated combinations using the two vertices of degree four and multiple copies of K_5.

4.3 Locally k-nested hamiltonian graphs

In this section I generalize the concepts introduced in the first section. The intuitive description of a locally k-nested hamiltonian graph G is that for any set of k mutually adjacent vertices $\{v_1, v_2, \ldots, v_k\}$ in $V(G)$, the induced graph on the neighbourhood of v_k in the neighbourhood of v_{k-1} in the neighbourhood of v_{k-2} in the neighbourhood of ... in the neighbourhood of v_1 is hamiltonian. A more compact formal definition is given below.

Definition 4.3.1. *For $k \geq 1$, a graph G is locally k-nested hamiltonian (L^kH) if for any subset $\{v_1, ..., v_k\}$ of k mutually adjacent vertices in G, $\langle N(v_1) \cap \cdots \cap N(v_k) \rangle$*

is a hamiltonian graph.

The definition for locally k-nested connected graphs is similar:

Definition 4.3.2. *For $k \geq 0$ a graph G is locally k-nested connected (L^kC) if for any subset $\{v_1, ..., v_k\}$ of k mutually adjacent vertices, $\langle N(v_1) \cap \cdots \cap N(v_k) \rangle$ is a connected graph. The case where $k = 0$ simply means the graph is connected.*

In the above definitions, the requirement that $\langle N(v_1) \cap ... \cap N(v_k) \rangle$ is a graph implies that it has at least one vertex (since the empty set is not a graph). This implies the following lemma.

Lemma 4.3.3. *If G is a graph that is L^mC for $m = 0, 1, ..., k$ and $n(G) \geq k + 2$, then every vertex $v \in V(G)$ lies in a $(k + 2)$-clique.*

Proof. The proof is by induction on k. If $k = 0$, then G is a connected graph of order at least 2, so every vertex of G lies in a K_2. Thus the result holds for $k = 0$. Now suppose $k \geq 1$ and let v be any vertex in G. Then, by the induction hypothesis, v lies in a $(k+1)$-clique X. Since G is connected and $n(G) \geq k+2$, there is a vertex in $G - V(X)$ that is adjacent to a vertex, say x_1, in X. Since G is LC, $\langle N(x_1) \rangle$ is connected, so there is a vertex in $(V(G) - V(X)) \cap N(x_1)$ that is adjacent to a vertex, say x_2, in $X - x_1$. Thus $N(x_1) \cap N(x_2)$ contains a vertex in $G - V(X)$. If $k = 1$, then X is contained in a 3-clique, so then the result is proved. If $k \geq 2$, then G is LLC, so then $\langle N(x_1) \cap N(x_2) \rangle$ is connected and hence there is a vertex in $(V(G) - V(X)) \cap N(x_1) \cap N(x_2)$ that is adjacent to a vertex, say x_3, in $X - \{x_1, x_2\}$. Carrying on in this manner, we eventually find k vertices $x_1, x_2, \ldots, x_k$ such that there is a vertex z in $(V(G) - V(X)) \cap N(x_1) \cap \cdots \cap N(x_k)$ that is adjacent to the only remaining vertex in $X - \{x_1, x_2, \ldots, x_k\}$. Then $\langle \{z\} \cup V(X) \rangle$ is a $(k+2)$-clique that contains v. □

The corollary follows immediately from the proof of Lemma 4.3.3.

Corollary 4.3.4. *If G is a graph that is L^mC for $m = 0, 1, ..., k$ and $n(G) \geq k + 2$, then any edge $uv \in E(G)$ lies in a $(k + 2)$-clique.*

I will now examine some of the implications of the definition for the structure of L^kH graphs that are L^mC for $m = 0, 1, ..., k - 1$.

Lemma 4.3.5. *If G is an L^kH graph that is L^mC for $m = 0, 1, ..., k-1$, then $\delta(G) \geq k+2$.*

Proof. From Lemma 4.3.3 and since an L^kH graph is also L^kC, it follows that every vertex $v \in V(G)$ lies in a $(k+2)$-clique and therefore there exist $k-1$ vertices $u_1, \ldots, u_k$ such that $N(v) \cap N(u_1) \cap \cdots \cap N(u_{k-1})$ is not an empty set. That implies that $\langle N(v) \cap N(u_1) \cap \cdots \cap N(u_{k-1}) \rangle$ is a hamiltonian graph and hence has order at least 3, and therefore $|N(v) \cap N(u_1) \cap \cdots \cap N(u_{k-1})| \geq 3$, and the result follows. □

The next two corollaries follow immediately.

Corollary 4.3.6. *The smallest L^kH graph that is L^mC for $m = 0, 1, ..., k-1$ is K_{k+3}.*

Corollary 4.3.7. *If G is an L^kH graph that is L^mC for $m = 0, 1, ..., k-1$ and $d(v) = k+2$ for some $v \in V(G)$, then $\langle N(v) \rangle \cong K_{k+2}$.*

Repeated application of the next theorem shows that Definition 4.3.1 is equivalent to the intuitive description of L^kH graphs. This theorem will also be used in some of the proofs that follow.

Theorem 4.3.8. *Let G be an L^kH graph that is L^mC for $m = 0, 1, ..., k-1$. Then for any $v \in V(G)$, $\langle N(v) \rangle$ is an $L^{k-1}H$ graph that is L^mC for $m = 0, 1, ..., k-2$.*

Proof. Since G is L^mC for $m = 0, 1, ..., k$ (because G is also L^kH), it follows from Lemma 4.3.3 that v lies in a $(k+2)$-clique. Let $\{u_1, u_2, \ldots, u_{k-1}\}$ be any set of $k-1$ mutually adjacent neighbours of v. Then $\langle N(v) \cap N(u_1) \cap \cdots \cap N(u_{k-1}) \rangle$ is hamiltonian, since G is L^kH. Since $\langle N_G(v) \cap N_G(u_1) \cap \cdots \cap N_G(u_{k-1}) \rangle = \langle N_{\langle N(v) \rangle}(u_1) \cap N_{\langle N(v) \rangle}(u_2) \cap \cdots \cap N_{\langle N(v) \rangle}(u_{k-1}) \rangle$, it is clear that $\langle N_{\langle N(v) \rangle}(u_1) \cap N_{\langle N(v) \rangle}(u_2) \cap \cdots \cap N_{\langle N(v) \rangle}(u_{k-1}) \rangle$ is hamiltonian. Hence $\langle N(v) \rangle$ is $L^{k-1}H$. Similarly, $\langle N(v) \rangle$ is L^mC for $m = 0, 1, \ldots, k-2$. □

Theorem 4.3.9. *If $k \geq 1$ and G is an L^kH graph that is L^mC for $m = 0, 1, ..., k-1$, then G is $(k+2)$-connected and locally $(k+1)$-connected.*

Proof. The proof is by induction on k. The result obviously holds for $k = 1$. Now let $k \geq 2$, and let $v \in V(G)$. Then by Theorem 4.3.8, $\langle N(v) \rangle$ is $L^{k-1}H$ and L^mC

for $m = 0, 1, \ldots, k-2$. Hence, by the induction hypothesis, $\langle N(v)\rangle$ is $(k+1)$-connected. Hence G is locally $(k+1)$-connected and therefore, by Theorem 3.1.4, G is $(k+2)$-connected. □

In order to deal with L^kH graphs we'll need a way to construct and manipulate such graphs for any value of k. The following construction, which is a generalization of triangle identification, provides the necessary tool.

Construction 4.3.10. *(K_{k+2}-identification) For $i = 1, 2$, let G_i be an L^kH graph that contains a $(k+2)$-clique X_i with $V(X_1) = \{v_1, v_2, \ldots, v_{k+2}\}$ and $V(X_2) = \{u_1, u_2, \ldots, u_{k+2}\}$. Furthermore, suppose that for each distinct k-clique Y_i in X_i, there is a Hamilton cycle of $\bigcap_{v\in V(Y_i)} N(v)$ that contains the edge $\langle X_i - V(Y_i)\rangle$. Create a larger graph G by identifying the vertices v_j and u_j, $j = 1, 2, \ldots, k+2$ to a single vertex w_j, and by retaining all the edges present in the original two graphs. Figure 4.1 illustrates the procedure for $k = 2$. We say that G is obtained from G_1 and G_2 by identifying suitable K_{k+2}'s.*

Theorem 4.3.11. *If two L^kH graphs G_1 and G_2 that are L^mC for $m = 0, 1, ..., k-1$ are combined using K_{k+2}-identification to form a larger graph G, then G is also an L^kH graph that is L^mC for $m = 0, 1, ..., k-1$.*

Proof. Let X_i be a $(k+2)$-clique in G_i, for $i = 1, 2$. Let $V(X_1) = \{v_1, v_2, \ldots, v_{k+2}\} \subset V(G_1)$, $V(X_2) = \{u_1, u_2, \ldots, u_{k+2}\} \subset V(G_2)$, and let $W = \{w_1, w_2, \ldots, w_{k+2}\}$ be the vertices in G obtained by identifying v_i with u_i, $i = 1, 2, \ldots, k+2$. Observe that if Z is a clique in G that contains a vertex in $G_1 - W$, then $\langle \bigcap_{a\in V(Z)} N(a)\rangle$ is contained in $V(G_1)$.

It is therefore only necessary to consider k-cliques in W. Let Z be a k-clique in W and let e be the edge $\langle V(W) - V(Z)\rangle$. Then, by the definition of K_{k+2}-identification, there is a Hamilton cycle C_i in $\langle \bigcap_{z\in Z} N_{G_i}(z)\rangle$ containing the edge e, for $i = 1, 2$. Let $C_1 = v_lPv_mv_l$ and $C_2 = u_lQu_mu_l$ where the end vertices of e are v_l and v_m in G_1 and u_l and u_m in G_2. Then $C = w_lPw_mQw_l$ is a Hamilton cycle of $\langle \bigcap_{z\in Z} N_G(z)\rangle$.

Similarly, when checking that G is L^mC, $m = 0, 1, 2, \ldots, k-1$, we need only consider m-cliques in W. For any m-clique in W with vertices $w_1, w_2, \ldots, w_m$, both $\langle N_{G_1}(v_1) \cap \cdots \cap N_{G_1}(v_m)\rangle$ and $\langle N_{G_2}(u_1) \cap \cdots \cap N_{G_2}(u_m)\rangle$ are connected. It then follows that $\langle N_G(w_1) \cap \cdots \cap N_G(w_m)\rangle$ is connected, since the vertices in W induce a complete graph. □

Note that K_{k+2}-identification of two L^qH, L^mC, $m = 0, 1, 2, \ldots, k-1$ graphs where $0 < q < k$ does not in general result in an L^qH graph. For example, the graph in Figure 4.3 (a) was constructed using multiple copies of K_5, and is L^2H and LC, but is not LH.

The following construction will be required for Theorem 4.3.26.

Construction 4.3.12. *(K_{k+2}-identification within a graph) Let G_a be an L^kH graph that, for $i = 1, 2$, contains disjoint $(k+2)$-cliques X_i with $V(X_1) = \{v_1, v_2, \ldots, v_{k+2}\}$ and $V(X_2) = \{u_1, u_2, \ldots, u_{k+2}\}$. Furthermore, suppose that for each distinct k-clique Y_i in X_i, there is a Hamilton cycle of $\bigcap_{v\in V(Y_i)} N(v)$ that contains the edge $X_i - V(Y_i)$. Finally, let $N(V(X_1) \cap N(V(X_2) = \emptyset$. Create graph G by identifying the vertices v_j and u_j, $j = 1, 2, \ldots, k+2$ to a single vertex w_j, and by retaining all the edges present in the original graph. We say that G is obtained from G_a by identifying suitable K_{k+2}'s within G_a.*

Corollary 4.3.13. *Let G_a be an L^kH graph that is L^mC for $m = 0, 1, ..., k-1$ and let G be a graph that was obtained by identifying suitable K_{k+2}'s within G_a. Then G is also an L^kH graph that is L^mC for $m = 0, 1, ..., k-1$.*

Proof. We use the same notation as in Construction 4.3.12. The argument used in the proof of Theorem 4.3.11 is directly applicable here as well, since $N(V(X_1)) \cap N(V(X_2)) = \emptyset$. □

Lemma 4.3.14. *Let G_1 be an L^kH graph that is L^mC for $m = 0, 1, ..., k-1$ and that contains a vertex v_1 such that $d(v_1) = k+2$. Then $\langle N(v_1)\rangle \cong K_{k+2}$ and v_1 can be used $k+2$ times in K_{k+2}-identification, once in combination with each of the $k+2$ distinct subsets of $k+1$ of its neighbours.*

Proof. Let $N_{G_1}(v_1) = \{v_2, v_3, \ldots, v_{k+3}\}$. Throughout this proof, the vertices $\{v_1, v_2, \ldots, v_{k+3}\}$ that are used in K_{k+2}-identification will retain their labels. Since $d(v_1) = k+2$, it follows from Corollary 4.3.7 that $\langle N_{G_1}(v_1)\rangle \cong K_{k+2}$. Let $G_2, G_3, \ldots, G_{k+3}$ be the graphs that will successively be used in K_{k+2}-identification to form the graphs $G_{1,2}, G_{1,2,3}, \ldots, G_{1,2,\ldots,k+3}$. Furthermore, without loss of generality, let G_i

be combined with $G_{1,2,\ldots,i-1}$ using the $(k+3)$-clique $\langle\{v_1, v_2, \ldots, v_{k+3}\} - \{v_i\}\rangle$, $i = 2, 3, \ldots, k+3$ to create the graph $G_{1,2,\ldots,i}$. First consider using K_{k+2}-identification to combine G_1 with G_2 to create the graph $G_{1,2}$ and let $\{u_1, u_2, \ldots, u_k\} = \{v_1, v_2, \ldots, v_{k+3}\} - \{v_2\} - \{v_l, v_m\}$, where $\{v_l, v_m\} \subset \{v_1, v_2, \ldots, v_{k+3}\} - \{v_2\}$, $l \neq m$. It suffices to show that in every K_{k+2}-identification step, the graph $\langle N(u_1) \cap N(u_2) \cap \cdots \cap N(u_k)\rangle$ has a Hamilton cycle that includes the edge $v_l v_m$. Since $N(u_1) \cap N(u_2) \cap \cdots \cap N(u_k) = \{v_2, v_l, v_m\}$, $\langle N(u_1) \cap N(u_2) \cap \cdots \cap N(u_k)\rangle \cong K_3$, and it clearly follows that the edge $v_l v_m$ is part of the Hamilton cycle in $\langle N(u_1) \cap N(u_2) \cap \cdots \cap N(u_k)\rangle$. Note that after the K_{k+2}-identification is done, the edge $v_l v_m$ in the Hamilton cycle in $\langle N(u_1) \cap N(u_2) \cap \cdots \cap N(u_k)\rangle$ is replaced by a path containing only vertices that originated from G_2. This argument applies to any choice of l and m.

We now proceed with the next K_{k+2}-identification, that between $G_{1,2}$ and G_3 to create the graph $G_{1,2,3}$, and continue in this manner. Consider the case in which we combine $G_{1,2,\ldots,i-1}$ with G_i to form the graph $G_{1,2,\ldots,i}$. This is done by identifying $\{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+3}\} \subset V(G_{1,2,\ldots,i-1})$ with vertices in $V(G_i)$. Without loss of generality let $\{u_1, u_2, \ldots, u_k\} = \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+3}\} - \{v_l, v_m\}$, where $\{v_l, v_m\} \subset \{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+3}\}$, $l \neq m$. Note that $N(v_i) \cap V(G_i) = \emptyset$ for all $i = 2, 3, \ldots, k+3$. It follows that $N(u_1) \cap N(u_2) \cap \cdots \cap N(u_k) = \{v_l, v_m, v_i\} \cup X \cup Y$, where $X \subset V(G_l)$, $Y \subset V(G_m)$ if $l, j < i$, respectively, and $X = \emptyset$ and $Y = \emptyset$ if $l, j > i$, respectively. It follows that in $\langle N(v_1) \cap N(u_1) \cap N(u_2) \cap \cdots \cap N(u_k)\rangle$ there are two paths connecting v_l and v_m: one path includes v_i, and the other path is the edge $v_l v_m$ (see Figure 4.5). Also, by inductive hypothesis the graph $\langle N(v_1) \cap N(u_1) \cap N(u_2) \cap \cdots \cap N(u_k)\rangle$ is hamiltonian. Therefore in $G_{1,2,\ldots,i}$, the K_{k+2} graph induced by $\{v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{k+3}\}$ is suitable for use in K_{k+2}-identification. □

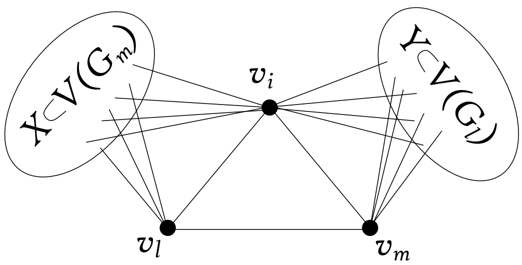

Figure 4.5: The graph $\langle N(u_1) \cap N(u_2) \cap \cdots \cap N(u_k)\rangle$ used in the proof of Lemma 4.3.14.

Remark 4.3.15. *A $(k+2)$-clique with vertices $v_1, v_2, \ldots, v_{k+2}$ in an L^kH graph G_1 can only be used once in K_{k+2}-identification to combine G_1 with an L^kH graph G_2. The reason is that before K_{k+2}-identification the edge $v_{k+1}v_{k+2}$ is part of a Hamilton cycle in $\langle N_{G_1}(v_1) \cap N_{G_1}(v_2) \cap \cdots \cap N_{G_1}(v_k)\rangle$. After K_{k+2}-identification, the edge $v_{k+1}v_{k+2}$ is replaced in the Hamilton cycle in $\langle N_G(v_1) \cap N_G(v_2) \cap \cdots \cap N_G(v_k)\rangle$ by a path with vertices that originated from G_2.*

At this point we have the necessary tools to start investigating the more interesting aspects of L^kH graphs that are L^mC for $m = 0, 1, \ldots, k-1$. I start with the relationship with k-trees.

Dirac [15] proved the following (the original formulation has been modified to bring it into line with the terminology used here):

Theorem 4.3.16. [15] *A graph G is a chordal graph if and only if every minimal cutset of G is a clique.*

From this we readily get the following corollary which will be required for the proof of Theorem 4.3.20.

Corollary 4.3.17. *If G is a k-tree, then G is a chordal graph.*

Rose [28] proved the following theorem that will be needed for the proof of Theorem 4.3.20.

Theorem 4.3.18. [28] *Let G be a k-tree and let u and v be any pair of nonadjacent vertices in G. Then there are exactly k vertex disjoint u - v paths in G.*

Observation 4.3.19. *If a given k-clique X is used r times ($r \geq 0$) in the construction of a k-tree G, then $G - V(X)$ has $r+1$ components, each of which contains one vertex of $\bigcap_{x \in V(X)} N(x)$.*

Theorem 4.3.20. *For $k \geq 3$ a k-tree G is an $L^{k-2}H$ graph that is L^mC for $m = 0, 1, \ldots, k-3$ if and only if G is a SC k-tree.*

Proof. First, suppose G is a k-tree that is not a SC k-tree. Then some k-clique X was used more than once in the k-tree construction of G. By Observation 4.3.19, there are three independent vertices u_1, u_2, u_3 in $\bigcap_{x \in V(X)} N_G(x)$. Now let Y be any $(k-2)$-clique in X and let $\{v_1, v_2\} = V(X) - V(Y)$. By Theorem 4.3.18, there are exactly k internally disjoint paths between any two vertices in $\{u_1, u_2, u_3\}$. Each such path contains exactly one vertex of X. Since $\{v_1, v_2\}$ are the only vertices of X in $\bigcap_{y \in V(Y)} N_G(y)$, any cycle in $\langle \bigcap_{y \in V(Y)} N_G(y) \rangle$ misses at least one of the vertices in $\{u_1, u_2, u_3\}$. Thus $\langle \bigcap_{y \in V(Y)} N_G(y) \rangle$ is not hamiltonian and hence G is not L^{k-2}-hamiltonian.

Now let G be a SC k-tree of order n. We prove by induction on n that G is $L^{k-2}H$. If $n = k + 1$, then $G = K_{k+1}$, which is obviously $L^{k-2}H$. Now assume $n \geq k + 2$. Let z be the last vertex added in the k-tree construction of G. Then $G - z$ is a SC k-tree of order $n - 1$ and $\langle N(z) \rangle$ is a k-clique in $G - z$ that has not been used in the k-clique construction of $G - z$. Let $N(z) = \{v_1, \ldots, v_k\}$. By Observation 4.3.19, $\langle \bigcap_{v \in N(z)} N_{G-z}(v) \rangle$ consists of a single vertex, say v_{k+1}. By our induction hypothesis, $G - z$ is $L^{k-2}H$. Thus, to prove that G is $L^{k-2}H$, we only need to show that the k-clique $\langle N(z) \rangle$ is suitable for k-clique identification.

Now consider any $(k-2)$-clique Y in $\langle N(z) \rangle$. Then $\langle \bigcap_{y \in V(Y)} N_{G-z}(y) \rangle$ has a Hamilton cycle C, since $G - z$ is $L^{k-2}H$. We may assume that $V(Y) = \{v_1, \ldots, v_{k-2}\}$. Then $\{v_{k-1}, v_k, v_{k+1}\} \subseteq \bigcap_{y \in Y} N_{G-z}(y)$ and v_{k+1} is the only common neighbour of v_{k-1} and v_k in $\bigcap_{y \in Y} N_{G-z}(y)$. Suppose C does not contain the edge $v_{k-1}v_k$. Then $\bigcap_{y \in Y} N_{G-z}(y)$ contains a $v_{k-1} - v_k$ path that contains neither the edge $v_{k-1}v_k$ nor the vertex v_{k+1}. Let P be a shortest such path. We note that v_{k-1} and v_k do not have a common neighbour on P, so P has at least four vertices and, by the minimality of P, the cycle $v_k v_{k-1} P v_k$ is chordless, contradicting Corollary 4.3.17. Hence C contains the edge $v_{k-1}v_k$, so $\langle N(z) \rangle$ is suitable for k-clique identification. This proves that G is $L^{k-2}H$. □

The proof of the next theorem will require the following lemma:

Lemma 4.3.21. *If G is an L^kH graph that is L^mC, $m = 0, 1, \ldots, k-1$ with $v \in V(G)$ and $n(G) \geq k + 4$, and $\langle N(v) \rangle$ is a complete graph, then $G - v$ is also an L^kH graph that is L^mC, $m = 0, 1, \ldots, k-1$.*

Proof. Only the neighbourhoods of vertices adjacent to v are affected by the removal of v from G, so to show that $G - v$ is L^kH, we need only consider the k-cliques that are contained in $\langle N(v) \rangle$. Let X be a k-clique in $\langle N(v) \rangle$. Then $\bigcap_{x \in V(X)} N_G(x)$ contains the vertex v and hence contains a Hamilton cycle C that contains a subpath $u_1 v u_2$, with $u_1, u_2 \in N(v) - V(X)$. Since $\langle N(v) \rangle$ is a complete graph, u_1u_2 is an edge in $\bigcap_{x \in V(X)} N_{G-v}(u)$. Replacing the path u_1vu_2 with the edge u_1u_2 yields a Hamilton cycle of $\bigcap_{x \in V(X)} N_{G-v}(u)$. Hence $G - v$ is L^kH.

It is also easily seen that $G - v$ is connected, and if $1 \leq m \leq k - 1$ and Z is any m-clique in $\langle N(v) \rangle$, then $\bigcap_{z \in V(Z)} N_{G-v}(v)$ is connected. Hence G is L^mC for $m = 0, 1, \ldots, k$. □

We note that a graph that is L^mH for $m = 1, 2, ..., k$ has minimum degree at least $k + 2$. Our next result follows from the fact that the neighbourhood of any vertex of degree $k + 2$ in such a graph induces a complete graph.

Corollary 4.3.22. *If G is a connected graph that is L^mH for $m = 1, 2, \ldots, k$, and a vertex v of degree $(k + 2)$ is removed from G, then $G - v$ is also an L^mH graph for $m = 1, 2, \ldots, k$.*

Theorem 4.3.23. *For each $k \geq 1$ there exists an L^kH graph that is L^mC for $m = 0, 1, ..., k - 1$ but that is not L^lH for $0 \leq l < k$ that has order $9 + 2k$. For each $k \geq 2$ there exists a nontraceable L^kH graph that is L^mC for $m = 0, 1, ..., k - 1$ but that is not L^lH for $0 \leq l < k$ that has order $10 + 2k$.*

Proof. By Theorem3.1.3 the smallest connected nonhamiltonian LH graph is of order 11, and in Theorem 4.2.7 we showed that if G is connected, LC, nonhamiltonian and L^2H, then $n(G) \geq 13$. From Theorem 3.4.6 we know that the smallest connected nontraceable LH graph is of order 14. Therefore the result holds for $k = 1$ and $k = 2$.

To prove the general case, we will show how to construct such graphs using K_{k+2}-identification. Combine two copies of K_{k+3} using K_{k+2}-identification. This results in an L^kH graph H_k that is L^mC for $m = 0, 1, ..., k - 1$ of order $k + 4$, that contains two vertices u and v of degree $k + 2$, and $u \not\sim v$. By Lemma 4.3.14, we can use $N[u]$ to combine H_k with $k + 2$ copies of K_{k+3}, and we can use $N[v]$ to add another three copies of K_{k+3} to create the L^kH graph G_k that is L^mC for $m = 0, 1, ..., k - 1$, where $n(G_k) = 9 + 2k$. Further note that G_k has a vertex cutset $V(H_k)$ of order $k + 4$, the removal of which breaks G_k into $k + 5$ components, meaning that G_k is not hamilonian. To create a nontraceable graph, use $N[v]$ to combine G_k with another copy of K_{k+3} to create the L^kH graph G'_k. Note that $n(G'_k) = 10 + 2k$ and that G'_k contains a vertex cutset of order $k+4$, the removal of which results in $k+6$ components, meaning that G'_k is not traceable. Figure 4.3 illustrates these graphs for $k=2$.

We still have to show G and G' are not $L^m H$, where $1 \leq m < k$. The two graphs G_2 and G'_2 are the graphs in Figure 4.3. In G_2 and G'_2, $\langle N(w) \rangle$ (where w is the vertex indicated in the figure) is the Goldner-Harary graph, which is the smallest nonhamiltonian LH graph. It follows that for $k=2$, G_k and G'_k are $L^k H$ but not $L^m H$ for $1 \leq m < k$. Using induction on k, assume that G_k and G'_k are $L^k H$ graphs that are $L^m C$ for $m = 0, 1, ..., k-1$ but not $L^m H$, where $1 \leq m < k$. In the subgraph H_k in both G_k and G'_k, let the graph induced by the $k+2$ vertices of degree $k+3$ be W_k. Add vertices u_1 and w_1 to G_k and G'_k to create the graphs F_{k+1} and F'_{k+1}. In F_{k+1} and F'_{k+1}, u_1 is adjacent to the vertices u and $V(W_k)$ and in F_{k+1}, w_1 is adjacent to all the vertices in $V(G_k)$ and in F'_{k+1}, w_1 is adjacent to all the vertices in $V(G'_k)$. Now F_{k+1} and F'_{k+1} are the graphs G_{k+1} and G'_{k+1} constructed above and $\langle N_{F_{k+1}}(w_1) \rangle$ is the graph G_k and $\langle N_{F'_{k+1}}(w_1) \rangle$ is the graph G'_k. See Figure 4.6 for an illustration of the technique.

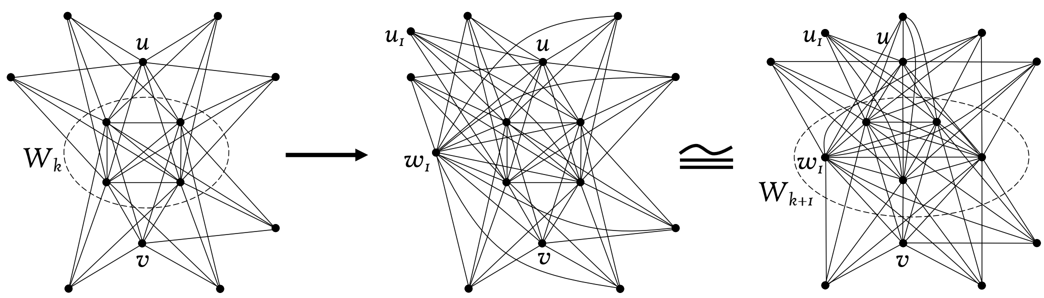

Figure 4.6: Converting an $L^2 H$ graph to an $L^3 H$ graph.

This completes the proof. □

I now turn my attention to minimum orders of connected nonhamiltonian graphs that are $L^m H$, $m = 1, 2, \ldots, k$.

Theorem 4.3.24. *For any $k \geq 1$, there exists a connected nonhamiltonian graph of order $9 + 2k$ that is $L^m H$ for every $m = 1, 2, \ldots, k$.*

Proof. A connected nonhamiltonian graph G_k of order $9 + 2k$ that is $L^m H$ for $m = 0, 1, ..., k$ can be constructed in the following way. Start with a K_{k+4} graph

W with $V(W) = \{w_0, w_1, \ldots, w_{k+3}\}$ and add a vertex u that is adjacent to all vertices in $V(W)$. Then add $k+4$ vertices v_i, $i = 0, 1, \ldots, k+3$, where $N(v_i) = \{w_i, w_{i+1}, \ldots, w_{i+k+1}\}$, where subscripts are taken modulo $k+4$. Figure 4.7 shows such a graph for $k = 2$. Graph $G11B$ in Figure 3.5 is the graph for $k = 1$.

To see that G_k is nonhamiltonian, note that $V(W)$ is a cutset, $|V(W)| < V(G)/2$ and $V(G) - V(W)$ is an independent set of vertices. It remains to be shown that G_k is $L^m H$, $m = 1, 2, \ldots, k$. The induced graphs on the neighbourhoods of each of $u, v_0, v_1, \ldots, v_{k+3}$ are complete graphs, and it follows that $\langle N(x_0) \cap \ldots \cap N(x_j) \rangle$ is hamiltonian, where $x_0 \in \{u, v_0, v_1, \ldots, v_{k+3}\}$ and $\{x_2, \ldots, x_j\} \subset N(x_0)$ and $j \leq k-1$.

To prove the result for the intersection of neighbourhoods of vertices in $V(W)$, we will use induction on k. It is easy to see that G_1 and G_2 meet the requirements of the theorem. Now assume that G_k is $L^m H$, $m = 1, 2, \ldots, k$. By inspection we find G_{k+1}, we find that $\langle N_{G_{k+1}}(w_1) \rangle \cong G_k - v_1$. It follows from Corollary 4.3.22 that $\langle N_{G_{k+1}}(w_1) \rangle$ is $L^m H$, $m = 1, 2, \ldots, k$. Also, $\langle N_{G_{k+1}}(w_1) \rangle$ is hamiltonian: $w_2 v_0 w_3 v_{k+4} w_4 v_3 w_5 v_4 \ldots w_{k+4} v_{k+3} w_0 u w_2$ is a Hamilton cycle.

Note that $\langle N(w_i) \rangle \cong \langle N(w_j) \rangle$, $i, j \in \{0, 1, \ldots, k+3\}$, so the result follows. □

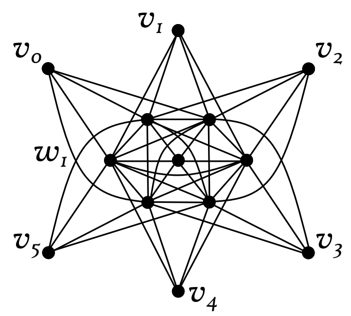

Figure 4.7: A connected graph of order 13 that is both LH and LLH but not hamiltonian.

In the light of Conjecture 4.1.2 it is interesting to note that the graphs constructed in the proof of Theorem 4.3.24 are locally $(k+1)$-connected, and contain an induced $K_{1,k+3}$, but as the proof of Corollary 4.3.25 makes clear, do not contain an induced $K_{1,k+4}$. Conjecture 4.1.2 is therefore the best possible, and the Oberly-Sumner conjecture is the best possible in a very strong sense.

Corollary 4.3.25. *For any $k \geq 1$ there exists a connected nonhamiltonian graph that is $L^m H$ for $m = 1, 2, \ldots, k$ that does not contain an induced $K_{1,k+4}$.*

Proof. Consider the graph G_k that is $L^m H$ for $m = 1, 2, \ldots, k$ constructed in the proof of Theorem 4.3.24. We use the same nomenclature as in the proof of Theorem 4.3.24. The vertex in a $K_{1,q}$ star that has degree greater than one is referred to as the *centre* vertex of the star. Since the neighbourhoods of the vertices $u, v_1, v_2, \ldots, v_{k+4}$ all induce complete graphs, it is clear that none of these vertices can be the centre vertex of an induced $K_{1,k+4}$. Since $\langle N(w_i)\rangle \cong \langle N(w_j)\rangle$ for $\{i, j\} \subseteq \{0, 1, \ldots, k+3\}$, we need only consider $\langle N(w_{k+3})\rangle$. $N(w_{k+3}) = \{w_0, w_1, \ldots, w_{k+2}, u, v_2, v_3, \ldots, v_{k+3}\}$. Since $\langle\{w_0, w_1, \ldots, w_{k+2}\}\rangle$ induces a complete graph, say W_{k+3}, and $w_i \sim u$, $i = 0, 1, \ldots, k+3$, and v_i, $i = 0, 1, \ldots, k+3$, only has neighbours in $V(W)$, it follows that $\alpha(\langle N(w_{k+3})\rangle) = k+3$, where α is the independence number. □

Similar constructions for connected nontraceable graphs that are $L^m H$ for $m = 1, 2, \ldots, k$ do not yield graphs of order $10 + 2k$, as is the case for nontraceable $L^k H$ graphs that are $L^m C$, $m = 0, 1, \ldots, k-1$, but rather graphs of order $12+2k$. This is because it is not possible to add another vertex of degree $k+2$ to the nonhamiltonian graph in such a way that the resulting graph is still $L^m H$ for $m = 1, 2, \ldots, k$. Figure 4.8 is an example of such a nontraceable graph that is LLH and LH of order 16. It is not known at this stage whether it is possible to improve on this result. It is speculated that this is due to these graphs being LH, since for connected LH graphs, the smallest nonhamiltonian graph has order 11 ($= 9+2k$), but the smallest nontraceable graph has order 14 ($= 12 + 2k$).

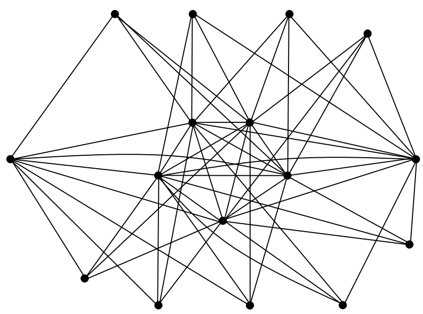

Figure 4.8: A nontraceable LH, LLH graph of order 16.

Next I investigate the complexity of the Hamilton Cycle Problem for $L^k H$ graphs.

I start with a theorem for L^2H graphs.

Theorem 4.3.26. *The Hamilton Cycle Problem for connected, locally connected L^2H graphs with maximum degree 12 is NP-complete.*

Proof. The proof will follow the same pattern as the proofs of Theorems 2.3.6 and 3.3.5. We start with a cubic graph G' and construct a connected, LC, L^2H graph G that is hamiltonian if and only if G' is hamiltonian.

Each vertex in G' is represented by a copy of K_5 in G, and will be referred to as a node in G.

Each edge in G' is represented by a more complex structure, that is based on the graph H in Figure 4.9. This is the graph that was constructed as part of the proof of Theorem 4.2.7 and is shown in Figure 4.3 (a) (it has been redrawn in Figure 4.9 to make it easier to represent the construction to follow). We use K_4-identification to combine H with two copies of graph D in Figure 4.9 in the following way: using the first copy of D we indentifty u_j and x_j, $j = 1, 2, 3, 4$, and using the second copy of D we identify v_j and x_j, $j = 1, 2, 3, 4$. This creates the graph F_i shown in Figure 4.10.

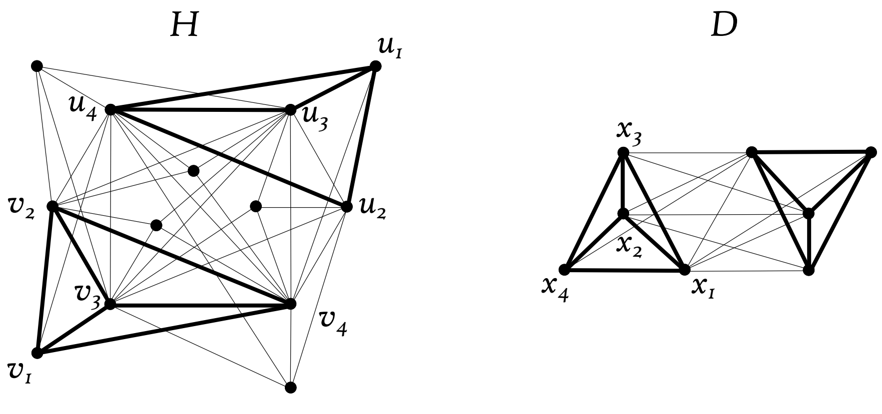

Figure 4.9: The graphs H and D used in the proof of Theorem 4.3.26.

The edges in G' are represented by copies of F_i in G, and will be referred to as "borders". The borders are connected to the nodes by means of K_4-identification. Let the vertices in a node in G be y_1, y_2, y_3, y_4, y_5 and let the vertices in F_i be labeled as shown in Figure 4.10. Since each vertex in G' has degree three, each node in G is attached to three copies of F_i. We identify the vertices as shown in Table 4.1

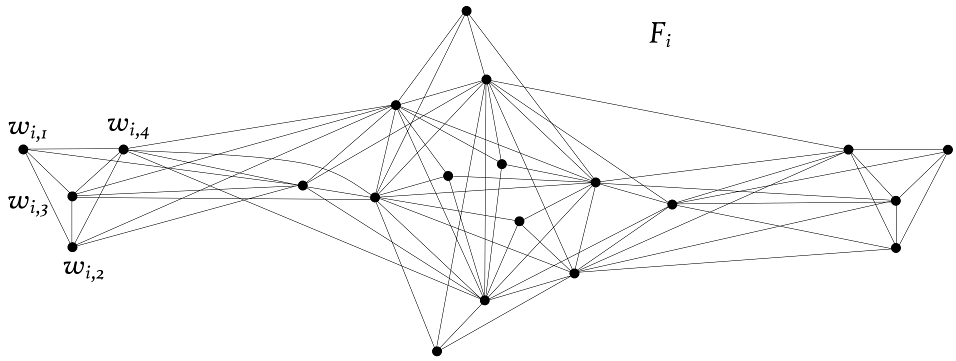

Figure 4.10: The graph F_i used in the proof of Theorem 4.3.26.

(after each vertex identification, the resulting vertex retains the y-label). We use the graphs F_1, F_2 and F_3 for illustrative purposes. See Figure 4.11 (the heavy lines in G represent edges belonging to the nodes).

Vertex in node	Vertex in F_i
y_1	$w_{1,2}$
y_2	$w_{1,1}$
y_4	$w_{1,4}$
y_5	$w_{1,3}$
y_1	$w_{2,3}$
y_2	$w_{2,2}$
y_3	$w_{2,1}$
y_5	$w_{2,4}$
y_1	$w_{3,1}$
y_2	$w_{3,2}$
y_3	$w_{3,3}$
y_4	$w_{3,4}$

Table 4.1: Vertices identified in the proof of Theorem 4.3.26.

Checking the degrees of the vertices that have been identified shows that $\Delta(G) = 12$ and by Theorem 4.3.11, Lemma 4.3.14 and Corollary 4.3.13, G is L^2H.

Figure 4.11 shows how a Hamilton cycle in G' can be translated to a Hamilton cycle in G (the heavy lines represent the Hamilton cycles). To see that if G is hamiltonian, then G' is also hamiltonian, consider the graph H that forms the connection between two nodes in G. Note that $u_2, u_3, u_4, v_2, v_3, v_4$ are the only neighbours of the five unlabeled vertices in Figure 4.9. Therefore any path cover of H contains at most one path that has one end vertex in u_1, u_2, u_3, u_4 and one end vertex in v_1, v_2, v_3, v_4. Thus every Hamilton cycle in G has at most one path from node Z_i to node Z_j that passes through the border between them. Since each node has three borders incident to it, the result follows.

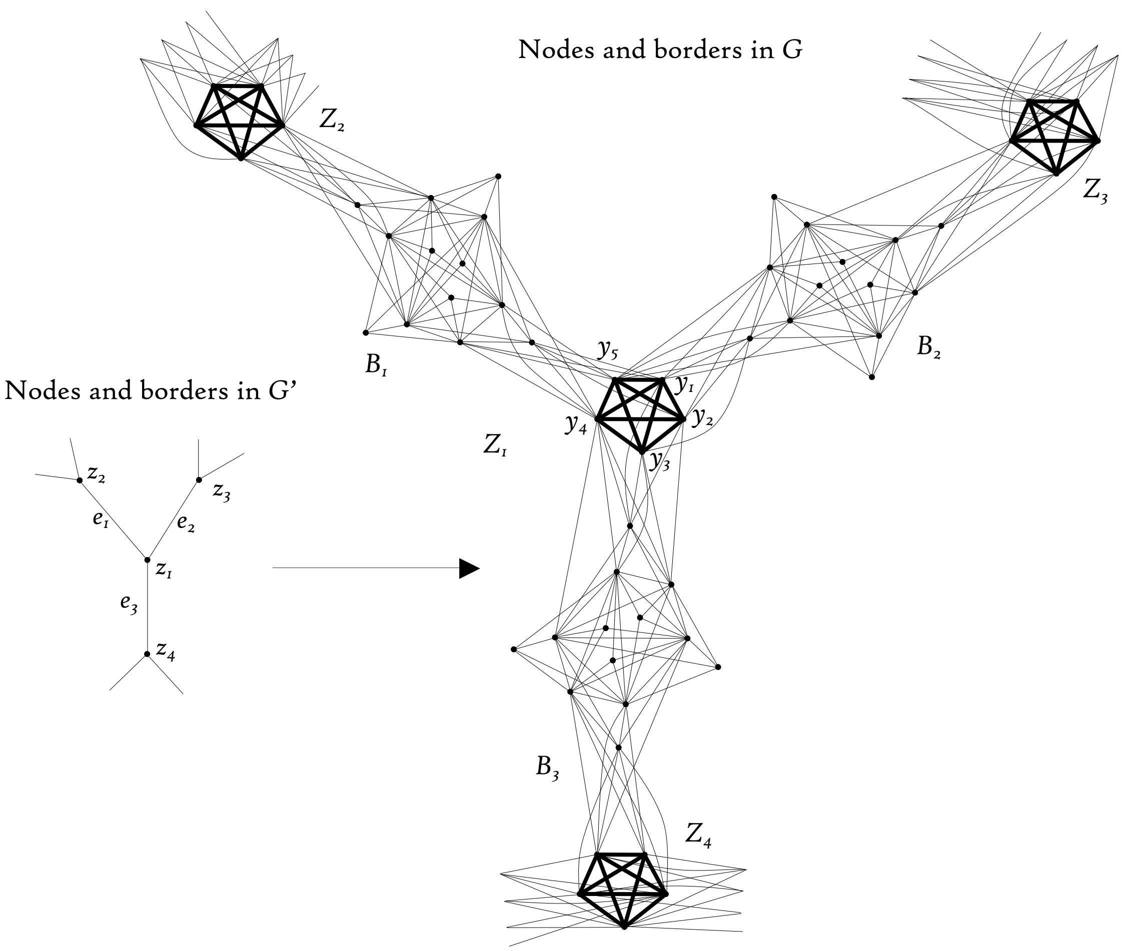

Figure 4.11: Converting the graph G' to the graph G in Theorem 4.3.26.

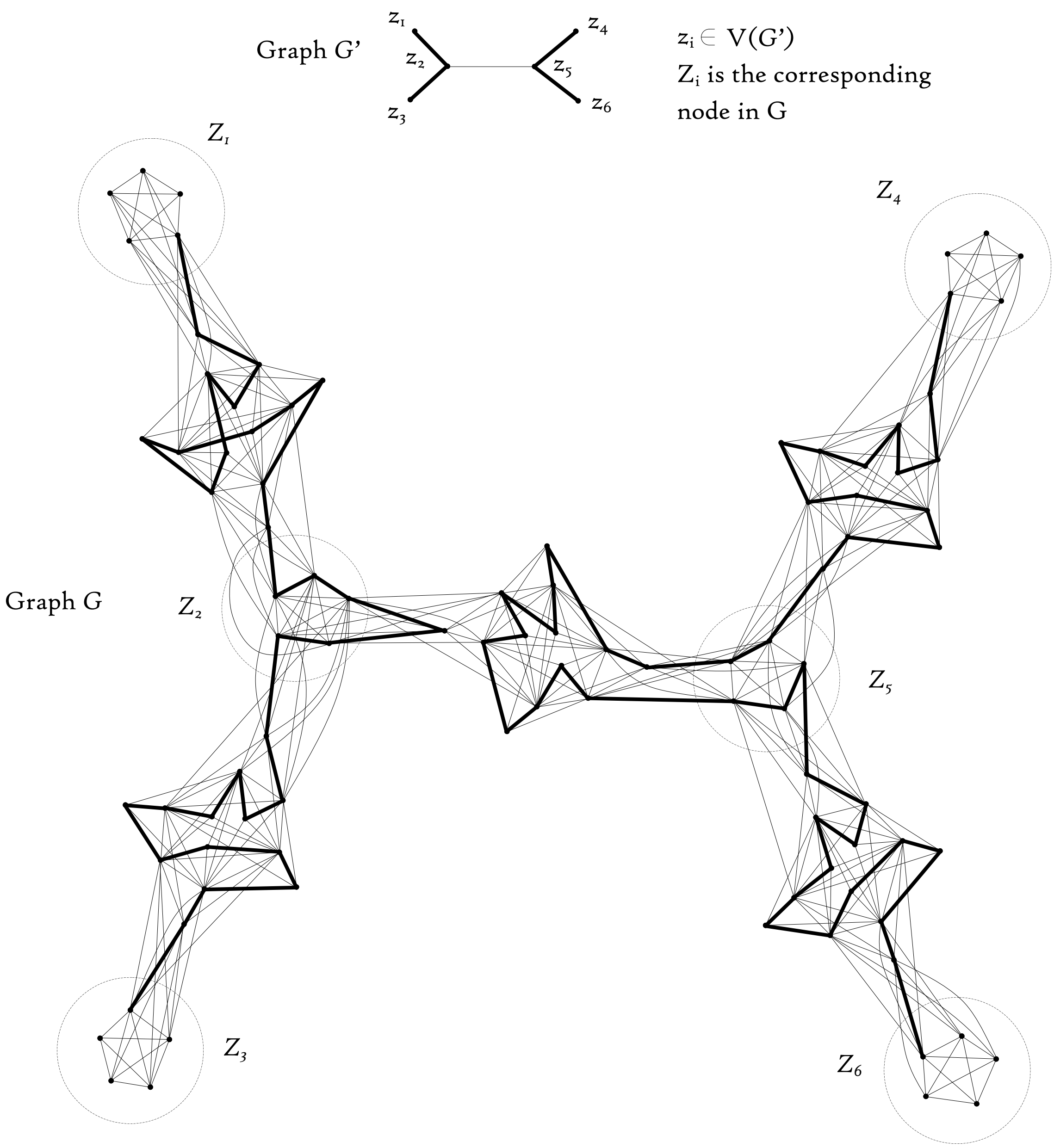

Figure 4.12: Translating a Hamilton cycle from G' to G in Theorem 4.3.26.

□

The proofs of Theorems 2.3.6, 3.3.5 and 4.3.26 rely on the existence of graphs that are LT, LH, or L^2H, respectively, and that have the following properties: they are nonhamiltonian, but traceable between two vertices of minimum degree, and if the order of the graph is $2q+1$, then the graph is $\frac{q}{q+1}$-tough. Note that the graphs of order $9+2k$ constructed in the proof of Theorem 4.3.23 have these properties for all values of k. It follows that similar NP-completeness theorems are possible for all $k \geq 3$ for graphs that are L^kH and L^mC for $m = 0, 1, ..., k-1$. The smallest value of the maximum degree that these theorems yield depends on the choice of neighbours for the vertices of minimum degree in the graphs of order $9 + 2k$. As k increases, there is increasing flexibility in the choice of neighbours for the vertices of minimum degree. Detailed calculations for $k = 3, 4, 5, 6, 7, 8$ show that the HCP for L^kH graphs that are L^mC for $m = 0, 1, ..., k-1$ is NP-complete for maximum degree $3k + 6$. When doing these calculations, the constructions follow a regular pattern and there is every reason to expect that the relationship $3k+6$ will hold for all $k \geq 1$.

When looking at the NP-completeness of the HCP for graphs that are L^mH for $m = 1, 2, \ldots, k$, we don't have the advantage of a theorem equivalent to Lemma 4.3.14. This means that any construction has to be checked in detail to confirm that the resulting graph is L^mH for $m = 1, 2, \ldots, k$. I begin with $k = 2$.

Theorem 4.3.27. *The HCP for graphs that are both LH and LLH with maximum degree 13 is NP-complete.*

Proof. We use the same construction as in the proof of Theorem 4.3.26, except that now the graph H is the graph shown in Figure 4.7. We combine H with two copies of the graph D to create the graph shown in Figure 4.13. When connecting borders to nodes to construct the graph G, we take care to limit the degree of vertices in the nodes to 10, as shown in Figure 4.14. Since the smallest connected nonhamiltonian LH graph has order 11, this ensures that in G, for any vertex v that lies in a node, $\langle N(v) \rangle$ is a hamiltonian graph. We still have to confirm that for any vertex u that is in a border and adjacent to a node, $\langle N(u) \rangle$ is hamiltonian. This is easily done, since there are only 8 such vertices in any border, and by symmetry, only one border has to be checked (see Figure 4.14). It follows that G is both LH and LLH.

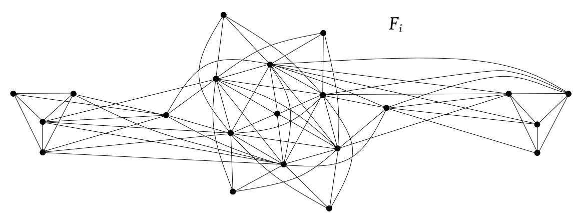

Figure 4.13: A border used in the construction of the graph G in Theorem 4.3.27.

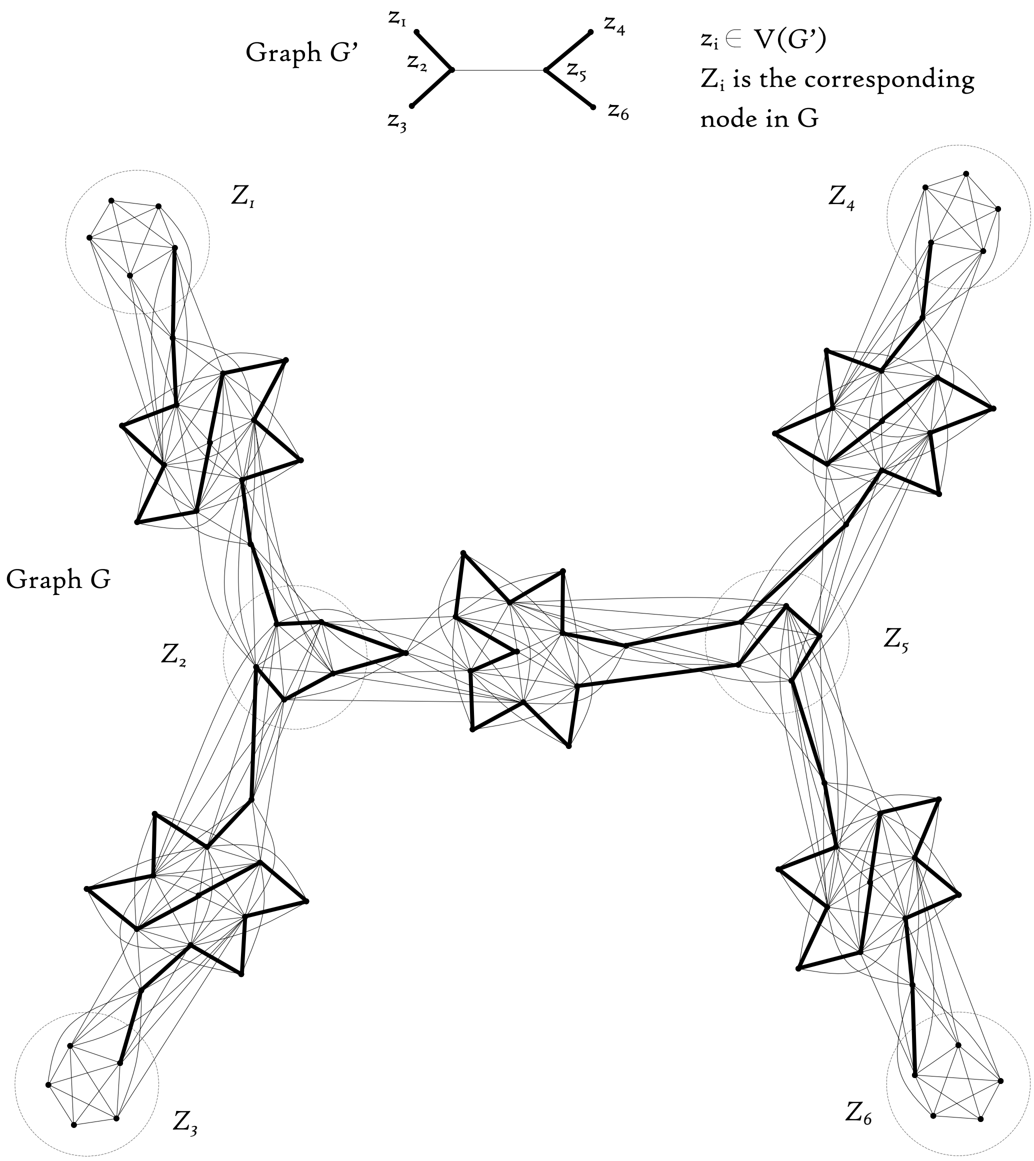

Figure 4.14: Translating a Hamilton cycle from G' to G in Theorem 4.3.27.

Again H has the properties discussed in the paragraph above this theorem, so we can assume that if G is hamiltonian then G' is hamiltonian. To see that G is hamiltonian if G' is hamiltonian, the reader is referred to Figure 4.14, where the heavy lines represent edges that are in a Hamilton cycle. □

Detailed calculations for the cases $k = 3$ and $k = 4$ show that the HCP is NP-complete for graphs that are $L^m H$ for $m = 1, 2, \dots, k$ that have maximum degree 16 for $k = 3$ and maximum degree 19 for $k = 4$. There appears to be a pattern according to which the HCP is NP-complete for graphs that are $L^m H$ for $m = 1, 2, \dots, k$ that have maximum degree $3k + 7$, for $k \geq 2$. Again there is reason to expect that the relationship will hold for all values of $k \geq 2$, since the pattern of the construction is quite regular. It is an interesting question whether these results are the best possible, particularly since for $k = 1$ we know the HCP is NP-complete for maximum degree $3k + 6$ (Theorem 3.3.5).

Finally, some additional properties of L^kH graphs will be derived.

Theorem 4.3.28. *For any $i \geq k + 2$ there exists a nontraceable L^kH graph G that is L^mC for $m = 0, 1, ..., k - 1$ such that $\delta(G) = i$.*

Proof. Starting with the nontraceable L^kH graph G'_k that is L^mC for $m = 0, 1, ..., k-1$ of order $10 + 2k$ constructed in the proof of Theorem 4.3.23, one can construct the graph G by using K_{k+2}-identification to combine G'_k with $k + 6$ copies of K_{i+1}, each time using a different vertex of degree $k + 2$ in G'_k. It is easy to arrange matters so that all vertices of degree higher than $k + 2$ in G'_k are used at least once in K_{k+2}-identification. To see that G is nontraceable, note that G contains a vertex cutset of $k + 4$ vertices (the $k + 4$ vertices of degree greater than $k + 2$ in G'_k), the removal of which breaks G into $k + 6$ components. □

It is already known that in a connected LH graph the detour order can be a vanishing fraction of the order of the graph ([16], Theorem 3.6.3). A similar result is possible for L^kH graphs that are L^mC for $m = 0, 1, ..., k - 1$. To prove this I will need the following two lemmas.

Lemma 4.3.29. *Let T_d be a tree of height d, such that all leaves are at height d, and all vertices that are not leaves have degree $r \geq 2$. Let T'_d be a subgraph of T_d obtained by starting at the root vertex and excluding one branch of T_d (and all its subbranches) at each vertex. Then $\lim_{d\to\infty} \frac{n(T'_d)}{n(T_d)} = 0$.*

Proof. Let the root vertex of T_d be v_0 and let the set of vertices in $V(T_d)$ at distance j from v_0 be $\{v_{j,1}, v_{j,2}, \ldots, v_{j,r^j}\}$. Then $n(T_d) = \sum_{i=0}^{d} r^i$ and $n(T'_d) = \sum_{i=0}^{d} (r-1)^i$. It follows that

$$\lim_{d\to\infty} \frac{n(T'_d)}{n(T_d)} = \lim_{d\to\infty} \frac{\sum_{i=0}^{d}(r-1)^i}{\sum_{i=0}^{d} r^i}$$
$$= 0$$

□

Corollary 4.3.30. *If each vertex of T_d in Lemma 4.3.29 is replaced by a connected graph of m vertices, the result still holds.*

Proof. This simply yields $\lim_{d\to\infty} \frac{n(T_d')}{n(T_d)} = \lim_{d\to\infty} \frac{m\sum_{i=0}^{d}(r-1)^i}{m\sum_{i=0}^{d}(r)^i}$ and the result follows as before. □

Theorem 4.3.31. *For $k > 1$, if G_n is an L^kH graph of order n that is L^mC, $m = 0, 1, \ldots, k-1$ with the smallest possible detour order D_n, then* $\lim_{n\to\infty} \frac{D_n}{n} = 0$.

Proof. We will show how to construct a family L^kH of graphs for which $\lim_{n\to\infty} \frac{D_n}{n} = 0$. Use K_{k+2}-identification to combine two copies of K_{k+3}, resulting in a graph H_k of order $k+4$. Let u, v be the two nonadjacent vertices in H_k. Then $\langle N(v)\rangle = \langle N(u)\rangle = \langle V(W_k)\rangle \cong K_{k+2}$, so each of u and v lies in $k+2$ distinct copies of K_{k+1}. Now use K_{k+2}-identification to add $2(k+2)$ vertices to $V(H_k)$ by combining H_k with $2(k+2)$ copies of K_{k+3} to create the graph X_k. This is done using each of the $k+2$ distinct sets of vertices that include u and $k+1$ of its neighbours, and the $k+2$ distinct sets of vertices that include v and $k+1$ of its neighbours. Label the vertices that have been added in this way $\{u_1, u_2, \ldots, u_{k+2}\}$ and $\{v_1, v_2, \ldots, v_{k+2}\}$ where $\{u_1, u_2, \ldots, u_{k+2}\} \subset N(u)$ and $\{v_1, v_2, \ldots, v_{k+2}\} \subset N(v)$. Now for each u_i $i = 1, 2, \ldots, k+2$, add $k+1$ vertices to the graph by successively combining the graph with $k+1$ copies of K_{k+3}, each time including u_i and the latest vertices that have been added in the set that is used for K_{k+2}-identification. This results in u_i lying in a K_{k+2}, call it U_i, that includes a vertex of minimum degree, which implies that U_i can be used in K_{k+2}-identification. The same is done for each v_i $i = 1, 2, \ldots, k+2$. The resulting graph is labeled G_0. Figure 4.15 shows G_0 for $k = 2$. From Theorem 4.3.11 if follows that G_0 is an L^kH graph that is L^mC, $m = 0, 1, \ldots, k-1$.

G_0 has vertex cutset $V(H_k)$ of order $k+4$, the removal of which results in a graph consisting of $U_1, U_2, \ldots, U_{k+2}, V_1, V_2, \ldots, V_{k+2}$, and none of these subgraphs is connected to any of the others. It follows that a longest path in G_0 can only include vertices from at most $k+5$ of these subgraphs. The graph G_1 is constructed from G_0 and and a further $2k+4$ copies of G_0, labeled $G_{0,1}, G_{0,2}, \ldots, G_{0,2k+4}$. The subgraphs in $G_{0,i}$, $i = 1, 2, \ldots, 2k+4$ are labeled in the same way as in G_0, except that the subscript will be preceded by the subscript of the graph. For instance, the subgraph in $G_{0,i}$ corresponding to U_4 in G_0 is labeled $U_{i,4}$. For $i = 1, 2, \ldots, k + 2$, identify U_i in G_0 with $U_{i,i}$ in $G_{0,i}$ and identify V_i in G_0 with $V_{i,i}$ in $G_{0,i+k+2}$. Since a longest path in G_0 can only include vertices from at most $k + 5$ of the elements of $U_1, U_2, \ldots, U_{k+2}, V_1, V_2, \ldots, V_{k+2}$, if follows that only vertices from $k + 5$ of the graphs $G_{0,1}, G_{0,2}, \ldots, G_{0,2k+4}$ can have vertices on any given longest path in G_1.

In the graph G_1 each $U_{i,j}$ and $V_{i,j}$, $i, j = 1, 2, \ldots, k + 2$, $i \neq j$ subgraph can be used to combine G_1 with another copy of G_0 to create the graph G_2. This process can continue indefinitely. This creates a tree-like structure, where each node is represented by a copy of H_k. Since $|H_k| = k + 4$ and each H_k is adjacent to $(2k + 4)$ K_{k+2} subgraphs (the subgraphs represented by $U_{j,i}$ and $V_{j,i}$), it follows that at each node of the tree-like structure a longest path in G_j misses $(2k+4)-(k+5)$ branches. From Corollary 4.3.30 the result follows. □

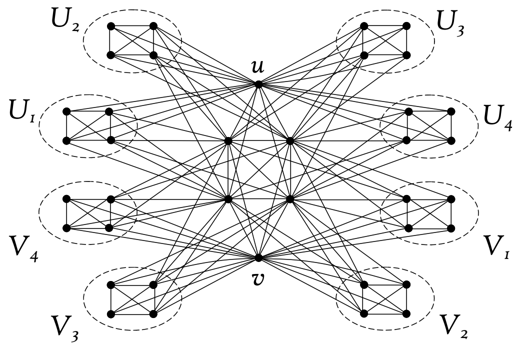

Figure 4.15: The graph G_0 used in Theorem 4.3.31.

The following two theorems are intuitively obvious, but are included for the record.

Theorem 4.3.32. *Let G be a nontraceable L^kH graph that is L^mC, $m = 0, 1, \ldots, k-1$ of order n with the smallest possible size S_n. Then* $\lim_{n\to\infty} \frac{S_n}{|E(K_n)|} = 0$.

Proof. Consider the order $10 + 2k$ nontraceable L^kH graph G' that is L^mC, $m = 0, 1, \ldots, k - 1$ constructed in the proof of Theorem 4.3.23. $|E(G')| = (k + 2)(k + 1)/2 + 2(k + 2) + (k + 6)(k + 2) = (k + 2)(3k + 17)/2$. Use K_{k+2}-identification to combine G' with copies of itself in a long chain to create the graph H_i where i is the number of copies of G' that have been combined. Then $|E(H_i)| = i(k + 2)(3k + 17)/2 - (i - 1)(k + 2)(k + 1)/2 = i(k + 2)(2k + 16)/2 + (k + 2)(k + 1)/2$, and

$V(H_i) = i(2k+10) - (i-1)(k+2) = i(k+8) + (k+2)$. So we have $\lim_{n\to\infty} \frac{|E(H_i)|}{|E(K_n)|} = \lim_{i\to\infty} \frac{i(k+2)(2k+16)+(k+2)(k+1)}{(i(k+8)+(k+2))(i(k+8)+(k+1))} = 0$. $\square$

Theorem 4.3.33. *Let G be a nontraceable L^kH graph that is L^mC, $m = 0, 1, \ldots, k-1$ of order n with the greatest possible size S_n. Then* $\lim_{n\to\infty} \frac{S_n}{|E(K_n)|} = 1$.

Proof. Consider the order $10 + 2k$ nontraceable L^kH graph G' that is L^mC, $m = 0, 1, \ldots, k-1$ constructed in the proof of Theorem 4.3.23. $|E(G')| = (k+2)(k+1)/2 + 2(k+2) + (k+6)(k+2) = (k+2)(3k+17)/2$ and $|V(G)| = 2k+10$. Use K_{k+2}-identification to combine G' with K_i to create the graph H. Then $|E(H)| = (k+2)(3k+17)/2 + i(i-1)/2 - (k+2)(k+1)/2 = (i^2 - i + 2k^2 + 20k + 32)/2$ and $|V(H)| = 2k + 10 + i - (k+2) = k + 8 + i$. So we have $\lim_{n\to\infty} \frac{|E(H)|}{|E(K_n)|} = \lim_{i\to\infty} \frac{(i^2-i+2k^2+20k+32)}{(i+k+8)(i+k+7)} = 1$. $\square$

Appendices

Appendix A

Theorem 3.3.4

Theorem *Let G be a connected nonhamiltonian LH graph of order $n = 12$. Then $\Delta(G) = 9$.*

Proof. Let w be a vertex in $V(G)$ of maximum degree Δ, and let $N(w) = \{v_1, v_2, \ldots, v_\Delta\}$, where the vertices are numbered such that $C = v_1 v_2 \ldots v_\Delta v_1$ is a Hamilton cycle in $\langle N(w) \rangle$. Let $X = \{x_1, x_2, \ldots, x_{12-\Delta-1}\}$ be the vertices not in $N[w]$. Until indicated to the contrary, assume that there are no edges between vertices in X.

We start by making some claims (note that if $|X| = 3$ then $\Delta = 8$). For convenience, the subgraphs forbidden by the claims to follow are shown in Figure A.1.

<u>Claim 1:</u> If $|X| = 3$, then if $\{v_i, v_{i+1}\} \subset N(x_k)$, if follows that $\{v_j, v_{j+1}\} \not\subset N(x_l)$, $i \neq j$, $k \neq l$, $k, l \in \{1, 2, 3\}$.

Proof of Claim 1: Let $v_1, v_2 \in N(x_1)$ and $v_i, v_{i+1} \in N(x_2)$, where $i \geq 2$ and let $v_k, v_l \in N(x_3)$, where $k, l \in \{1, 2, \ldots, \Delta\}$, $l \neq k$. There are two cases to consider.

Case1. If $k \in \{2, 3, \ldots, i\}$ and $l \in \{1, i+1, i+2, \ldots, \Delta\}$. Then $v_1 x_1 v_2 v_3 \ldots v_{k-1} w v_{l-1} v_{l-2} \ldots v_{i+1} x_2\ v_i v_{i-1} \ldots v_k x_3 v_l v_{l+1} \ldots v_\Delta v_1$ is a Hamilton cycle in G.

Case 2. If $k, l \in \{2, 3, \ldots, i\}$ then $v_1 x_1 v_2 v_3 \ldots v_k x_3 v_l v_{l-1} \ldots v_{k+1} w\ v_{l+1} v_{l+2} \ldots v_i x_2\ v_{i+1} v_{i+2} \ldots v_\Delta v_1$ is a Hamilton cycle in G (here we assumed $l > k$).

By symmetry, and since $\delta(G) \geq 3$, the result follows.

<u>Claim 2:</u> If $\Delta(G) \leq 8$ then $|N(v_i) \cap X| \leq 2$, $i \in \{1, 2, \ldots, 8\}$.

Proof of Claim 2: Let $\{x_1, x_2, x_3\} \subset N(v_i)$. Since $\{x_1, x_2, x_3\}$ is an independent set, a Hamilton cycle in $\langle N(v_i) \rangle$ contains at least four vertices in $N(w) \cap N(v_i)$. $\Delta(G) \leq 8$ implies that $\Delta(G) = 8$ and that say, $x_1 \in N(v_{i-1})$ and say, $x_2 \in N(v_{i+1})$, which is counter to Claim 1.

Claim 3: If $|X| = 3$, $\{v_i, v_{i+1}\} \subset N(x_1)$, and $v_j \in N(x_2)$, $i \neq j$, then $v_{j+1} \notin N(x_3)$.

Proof of Claim 3: Without loss of generality let $\{v_\Delta, v_1\} \subset x_1$, let $x_2 \sim v_i$ and $x_3 \sim v_{i+1}$, $i \neq \Delta$, and let $v_k \sim x_2$ and let $v_l \sim x_3$.

We know from Claim 1 that $x_2 \not\sim v_{i+1}$ and $x_3 \not\sim v_i$. By symmetry there are three cases to consider.

Case 1: $k \in \{1, 2, \ldots, i-1\}$ and $l \in \{i+2, i+3, \ldots, \Delta\}$. Then $v_1v_2 \ldots v_kx_2v_iv_{i-1} \ldots v_{k+1}wv_{l-1}v_{l-2} \ldots v_{i+1}x_3v_lv_{l+1} \ldots v_\Delta x_1v_1$ is a Hamilton path in G.

Case 2: $k, l \in \{i+2, i+3, \ldots, v_\Delta\}$, $k > l$. Then $v_1v_2 \ldots v_ix_2v_kv_{k-1} \ldots v_lx_3v_{i+1}v_{i+2} \ldots v_{l-1}wv_{k+1}v_{k+2} \ldots v_\Delta x_1v_1$ is a Hamilton path in G.

Case 3: $k, l \in \{i+2, i+3, \ldots, v_\Delta\}$, $l > k$. Then $v_1v_2 \ldots v_ix_2v_kv_{k+1} \ldots v_lx_3v_{i+1}v_{i+2} \ldots v_{k-1}wv_{l+1}v_{l+2} \ldots v_\Delta x_1v_1$ is a Hamilton path in G.

Claim 4: If $|X| = 3$, then $x_jv_ix_kv_{i+1}x_l$, $j, k, l \in \{1, 2, 3\}$, $j \neq k \neq l$, is not a path in G.

Proof of Claim 4: Without loss of generality let $x_1v_1x_2v_2x_3$ be a path in G. By Claim 1, $x_1 \not\sim v_8$ and $x_3 \not\sim v_3$. If $x_1 \sim v_3$, then $v_3x_1v_1x_2v_2x_3$ is a path in G, and since $N[w]$ is traceable between any two elements of $N(w)$ and $\delta(G) \geq 3$, G is Hamiltonian. Similarly, $x_3 \not\sim v_8$. Therefore x_1 and x_3 each have at least two neighbours in $\{v_4, v_5, v_6, v_7\}$ and it follows from Claims 1 and 3 that x_1 and x_3 have the same two neighbours in $\{v_4, v_5, v_6, v_7\}$. By symmetry we may assume the neighbours are either v_4 and v_6 or v_4 and v_7, so that we may assume that $v_4 \sim x_1$ and $v_4 \sim x_3$. But x_2 also has at least one additional neighbour. If $x_2 \sim v_3$, $v_1v_8v_7v_6v_5wv_3x_2v_2x_3v_4x_1v_1$ is a Hamilton cycle in G. If $x_2 \sim v_5$, $v_1x_1v_4x_3v_2v_3wv_8v_7v_6v_5x_2v_1$ is a Hamilton cycle in G. If $x_2 \sim v_8$, $v_1x_1v_4x_3v_2v_3wv_5v_6v_7v_8x_2v_1$ is a Hamilton cycle in G. Therefore x_2 must be adjacent to either v_6 or v_7. If $x_2 \sim v_6$, then $x_1 \sim v_7$ and $x_3 \sim v_7$ by Claim 2. Then $v_1v_8v_7x_1v_4x_3v_2v_3wv_5v_6x_2v_1$ is a Hamilton cycle in G. If $x_2 \sim v_7$, then $x_1 \sim v_6$ and $x_3 \sim v_6$ and $v_1v_8v_7x_2v_2v_3wv_5v_4x_3v_6x_1v_1$ is a Hamilton cycle in G. This completes the proof of Claim 4.

Claim 5: If $|X| = 3$, then it is not possible that both $\{v_i, v_{i+1}, v_{i+2}\} \subset N(x_j)$ and $\{v_i, v_{i+2}\} \subset N(x_k)$, $j \neq k$.

Proof of Claim 5: Without loss of generality let $\{v_1, v_2, v_3\} \subset N(x_1)$ and $\{v_1, v_3\} \subset$

$N(x_2)$. By Claim 2, $x_3 \not\sim v_1$ and $x_3 \not\sim v_3$. By Claim 3, $x_3 \not\sim v_2$, $x_3 \not\sim v_8$ and $x_3 \not\sim v_4$. Since $\delta(G) \geq 3$, $\{v_5, v_6, v_7\} \subset N(x_3)$, but that is against Claim 1.

Claim 6: If $|X| = 3$, then it is not possible that both $\{v_i, v_{i+1}, v_{i+2}\} \subset N(x_j)$ and $\{v_i, v_{i+3}\} \subset N(x_k)$, $j \neq k$.

Proof of Claim 6: Without loss of generality let $\{v_1, v_2, v_3\} \subset N(x_1)$ and $\{v_1, v_4\} \subset N(x_2)$. By Claim 2, $x_3 \not\sim v_1$ and and by Claim 3, $x_3 \not\sim v_2$, $x_3 \not\sim v_3$, and $x_3 \not\sim v_8$. Since $\delta(G) \geq 3$, x_3 has 3 neighbours in $\{v_4, v_5, v_6, v_7\}$, but that is against Claim 1.

Claim 7: If $|X| = 3$, then it is not possible that $\{v_i, v_{i+1}, v_{i+2}\} \subset N(x_j)$, $v_{i+1} \in N(x_k)$, and $v_{i+3} \in N(x_l)$ $j \neq k \neq l$.

Proof of Claim 7: Without loss of generality let $\{v_1, v_2, v_3\} \subset N(x_1)$, $v_2 \sim x_2$, and $v_4 \sim x_3$. By Claim 1, $x_3 \not\sim v_3$ and $x_3 \not\sim v_5$, by Claim 2, $x_3 \not\sim v_2$, and by Claim 3 $x_3 \not\sim v_1$. Therefore by Claim 1, $x_3 \sim v_6$ and $x_3 \sim v_8$. By Claim 1 $x_2 \not\sim v_1$ and $x_2 \not\sim v_3$, by Claim 3 $x_2 \not\sim v_5$ and $x_2 \not\sim v_7$, so x_2 is adjacent to two vertices in $\{v_4, v_6, v_8\}$. If $x_2 \sim v_4$, $v_1v_2x_2v_4v_5v_6x_3v_8v_7wv_3x_1v_1$ is a Hamilton cycle in G and if $x_2 \sim v_6$, $v_1v_2x_2v_6v_5v_4x_3v_8v_7wv_3x_1v_1$ is a Hamilton cycle in G. The claim follows.

Claim 8: If $|X| = 3$, then it is not possible that both $\{v_i, v_{i+2}\} \subset N(x_j)$ and $\{v_{i+1}, v_{i+3}\} \subset N(x_k)$, $j \neq k$.

Proof of Claim 8: Without loss of generality let $\{v_8, v_2\} \subset N(x_1)$ and $\{v_1, v_3\} \subset N(x_2)$. By Claim 3 x_3 is not adjacent to at least one of v_1 and v_2, so x_3 must be adjacent to at least two vertices in $\{v_3, v_4, v_5, v_6, v_7, v_8\}$, say v_i and v_j, $i < j$. Then a Hamilton cycle can be found: $v_8x_1v_2v_1x_2v_3v_4 \ldots v_i\ x_3v_jv_{j-1} \ldots v_{i+1}wv_{j+1}v_{j+2} \ldots v_8$.

We will now systematically work our way through the possible graphs for which $\Delta(G) = 8$ and $|X| = 3$, incrementing first the neighbours of x_3, then the neighbours of x_2, and lastly the neighbours of x_1. So we start with x_1 being adjacent to v_1, v_2 and v_3, and x_2 being adjacent to v_1. For the sake of brevity, the claims will only be referred to by their numbers, so for example, Claim 1 will be referred to simply as (1). Each iteration will be headed by the edges between X and $N(w)$ that are assumed to be in G in that iteration. Note that if $x_i \sim v_j$ is specified in the header of the iteration, then $v_k \notin N(x_i)$ if $k < j$ unless such edges are also explicitly specified in the header of the iteration.

$\{v_1, v_2, v_3\} \subset N(x_1)$, $x_2 \sim v_1$. Note that $v_2, v_3, v_4, v_8 \notin N(x_2)$ by (1), (5) and (6) and $v_1, v_2, v_8 \notin N(x_3)$ by (2), (4) and (3). So by (1) we have $N(x_3) = \{v_3, v_5, v_7\}$.

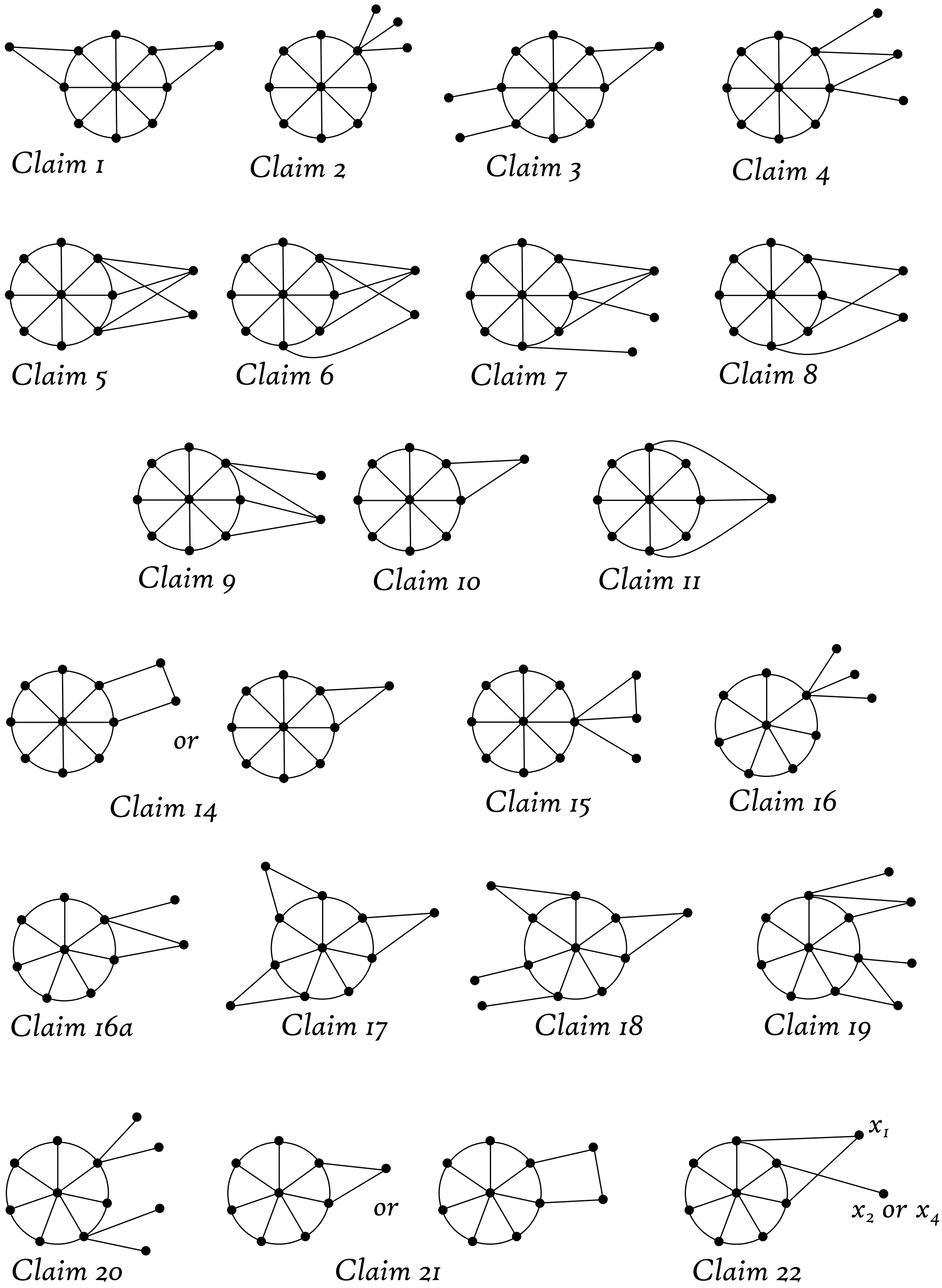

Figure A.1: Forbidden subgraphs according to the claims in the proof of Theorem 3.3.4. Note that for Claim 16a the claim is somewhat different: the subgraph is not forbidden. There are no sketches for Claims 12 and 13.

Then by (3) $v_6 \notin N(x_2)$, so $N(x_2) = \{v_1, v_5, v_7\}$. Also, $v_4, v_5, v_7, v_8 \notin N(x_1)$ by (2) and (8). If $v_6 \sim x_1$, then $v_1v_2x_1v_6v_5x_2v_7x_3v_3v_4wv_8v_1$ is a Hamilton cycle in G, so $N(x_1) = \{v_1, v_2, v_3\}$. Now, since $\langle N(v_1)\rangle$ is hamiltonian and $d(x_1) = d(x_2) = 3$, and $\Delta(G) = 8$, it follows that $\{v_2, v_3, v_5, v_7, v_8\} = N(v_1) \cap N(w)$. When we consider $\langle N(v_5)\rangle$, by a similar argument we find that $\{v_1, v_3, v_4, v_6, v_7\} \subset N(v_5)$. Note that if the edge v_4v_6 is added the graph becomes hamiltonian: $v_1v_2x_1v_3v_4v_6wv_8v_7x_3v_5x_2v_1$. Therefore a Hamilton cycle in $\langle N(v_5)\rangle$ must include the path $v_3x_3v_7x_2v_1w$. It is then clear that it is not possible to extend the path to include both v_4 and v_6 and end at v_3. Therefore $\langle N(v5)\rangle$ is not hamiltonian, and the case is not possible.

We have now proved Claim 9: If $|X| = 3$ and $\{v_i, v_{i+1}, v_{i+2}\} \subset N(x_j)$, then $v_i \notin N(x_k)$, $k \neq j$.

$\{v_1, v_2, v_3\} \subset N(x_1)$, $x_2 \sim v_2$. By (9) $v_1, v_3 \notin N(x_2)$ and $v_1, v_3 \notin N(x_3)$. Also, $x_3 \not\sim v_2$ by (2), $x_3 \not\sim v_8$ and $x_3 \not\sim v_4$ by (7). Therefore $N(x_3) = \{v_5, v_6, v_7\}$, which is counter to (1).

By (9) the next iteration to consider is

$\{v_1, v_2, v_3\} \subset N(x_1)$, $x_2 \sim v_4$. By previous cases, $N(x_2) \cup N(x_3) \subset \{v_4, v_5, v_6, v_7, v_8\}$. Therefore by (1), $N(x_2) = N(x_3) = \{v_4, v_6, v_8\}$. It follows that $\langle N(v_6)\rangle$ is not hamiltonian, since $|N(v_6) \cap (N(x_2) \cap N(x_3))| \leq 2$.

By (1) the next iteration to consider is

$\{v_1, v_2, v_4\} \subset N(x_1)$, $x_2 \sim v_1$. Now $v_1, v_2, v_8 \notin N(x_3)$ by (2), (4) and (3), so by (1), $N(x_3) = \{v_3, v_5, v_7\}$. Then $v_2, v_4, v_6, v_8 \notin N(x_2)$ by (3), so that x_2 must have two neighbours in $\{v_3, v_5, v_7\}$. But if $x_2 \sim v_3$, then $v_1v_2x_1v_4v_5v_6wv_8v_7x_3v_3x_2v_1$ is a Hamilton cycle in G, and if $x_2 \sim v_5$, then $v_1v_2x_1v_4v_3x_3\ v_7v_8wv_6v_5x_2v_1$ is a Hamilton cycle in G.

$\{v_1, v_2, v_4\} \subset N(x_1)$, $x_2 \sim v_2$. Now $v_1, v_2, v_3 \notin N(x_3)$ by previous case, (2) and (3), so by (1) $N(x_3) = \{v_4, v_6, v_8\}$. Then $v_3, v_4, v_5, v_7 \notin N(x_2)$ by (1), (2) and (3) so $N(x_2) = \{v_2, v_6, v_8\}$. Note that $v_3, v_6, v_7, v_8 \notin N(x_1)$ by (9), (2) and (8). Also, $x_1 \not\sim v_5$, otherwise $v_1x_1v_5wv_7v_6x_3v_4v_3v_2x_2v_8v_1$ is a Hamilton cycle in G. Therefore $N(x_1) = \{v_1, v_2, v_4\}$. So, if $\langle N(v_4)\rangle$ is hamiltonian, then $\{v_1, v_2, v_6, v_8\} \subset N(v_4)$, which implies that $\{v_1, v_2, v_3, v_5, v_6, v_8, x_1, x_2, w\} \subset N(v_4)$, so that $d(v_4) \geq 9$.

$\{v_1, v_2, v_4\} \subset N(x_1)$, $x_2 \sim v_3$. Now $x_2 \not\sim v_4$ by (1), and $x_3 \not\sim v_4$ by (3), therefore by (1), $x_3 \sim v_3$. So by (1) and (3), x_2 and x_3 must share the same two neighbours in

$\{v_5, v_6, v_7, v_8\}$. If the shared neighbours are v_5 and v_7, then $v_1v_2x_1v_4v_3x_2v_5x_3v_7v_6wv_8v_1$ is a Hamilton cycle in G. If the shared neighbours are v_5 and v_8, then $v_1v_2v_3x_2v_5x_3v_8v_7v_6wv_4x_1v_1$ is a Hamilton cycle in G. If the shared neighbours are v_6 and v_8, then $v_1v_2v_3x_2v_6x_3v_8v_7wv_5v_4x_1v_1$ is a Hamilton cycle in G. Therefore this case is not possible.

$\{v_1, v_2, v_4\} \subset N(x_1)$, $x_2 \sim v_4$. Since by earlier cases, $v_1, v_2, v_3 \notin N(x_2)$ and $v_1, v_2, v_3 \notin N(x_3)$, by (3) both $x_2 \sim v_4$ and $x_3 \sim v_4$, but that is contrary to (2).

By (1) the next iteration to consider is

$\{v_1, v_2, v_5\} \subset N(x_1)$, $x_2 \sim v_1$. Now $v_1, v_2, v_8 \notin N(x_3)$ by (2), (4) and (3). Therefore $N(x_3) = \{v_3, v_5, v_7\}$ by (3). Then $v_4, v_5, v_6, v_8 \notin N(x_2)$ by (3), (2) and (1), so that $N(x_2)$ must have two vertices in $\{v_2, v_3, v_7\}$, so by (1), $x_2 \sim v_7$. If $x_2 \sim v_2$, then $v_1v_8v_7v_6wv_4v_3x_3v_5x_1v_2x_2v_1$ is a Hamilton cycle in G. Therefore $N(x_2) = \{v_1, v_3, v_7\}$. Note by an earlier case, $x_1 \not\sim v_3$ and $x_1 \not\sim v_4$. Also, $x_1 \not\sim v_8$ by (8), and if $x_1 \sim v_6$, then $v_1v_2x_1v_6v_5v_4wv_8v_7x_3v_3x_2v_1$ is a Hamilton cycle in G. Therefore $N(x_1) = \{v_1, v_2, v_5\}$. Now, since $\langle N(v_5)\rangle$ is hamiltonian, it must be the case that $\{v_1, v_2, v_3, v_7\} \subset N(v_5)$, which implies $d(v_5) \geq 9$.

$\{v_1, v_2, v_5\} \subset N(x_1)$, $x_2 \sim v_2$. Now $v_1, v_2, v_3 \notin N(x_3)$ by (2) and (3). Therefore by (1), $N(x_3) = \{v_4, v_6, v_8\}$. By (1) and (3) $v_3, v_5, v_7 \notin N(x_2)$. So $N(x_2)$ contains two vertices in $\{v_4, v_6, v_8\}$. If $x_2 \sim v_4$, then $v_1v_2x_2v_4v_3wv_7v_8x_3v_6v_5x_1v_1$ is a a Hamilton cycle in G. If $x_2 \sim v_6$, then $v_1v_2x_2v_6v_7v_8x_3v_4v_3\ wv_5x_1v_1$ is a Hamilton cycle in G. So this case is not possible.

$\{v_1, v_2, v_5\} \subset N(x_1)$, $x_2 \sim v_3$. Now x_2 and x_3 are not adjacent to v_1 or v_2 by earlier cases, and $x_2 \not\sim v_4$ by (1) and $x_3 \not\sim v_4$ by (3), and by (3) we then get that $x_3 \sim v_3$. It follows by (1) and (3) that x_2 and x_3 must both have the same two neighbours in $\{v_5, v_6, v_7, v_8\}$, and since $x_1 \sim v_5$, x_2 and x_3 must both be adjacent to v_6 and v_8. But then $v_1v_2v_3x_2v_6x_3v_8v_7wv_4v_5x_1v_1$ is a Hamilton cycle in G.

$\{v_1, v_2, v_5\} \subset N(x_1)$, $x_2 \sim v_4$. By earlier cases and (1) and (3), $N(x_2) = N(x_3) = \{v_4, v_6, v_8\}$. But then $v_1v_2v_3v_4x_2v_6x_3v_8v_7wv_5x_1v_1$ is a Hamilton cycle in G.

By (1), (3) and symmetry, this exhausts the possibilities where x_1 has two successive neighbours in $N(w)$. So for the remainder of this part of the proof, it can be assumed that no vertex in X has two successive neighbours in $N(w)$. We'll refer to this as Claim 10. By (10) the next iteration is

$\{v_1, v_3, v_5\} \subset N(x_1)$, $\{v_1, v_3, v_5\} \subset N(x_2)$. By (8), x_3 is not adjacent to more than one of v_2 and v_4, so by (2) and (10), $x_3 \sim v_6$ and $x_3 \sim v_8$ and x_3 is adjacent to one of v_2 and v_4. If $x_3 \sim v_4$, then $v_1v_2wv_4x_3v_8v_7v_6v_5x_1v_3x_2v_1$ is a Hamilton cycle in G. By symmetry, $x_3 \not\sim v_2$, implying that $d(x_3) \leq 2$.

$\{v_1, v_3, v_5\} \subset N(x_1)$, $\{v_1, v_3, v_6\} \subset N(x_2)$. Note that if $x_3 \sim v_2$, then by (8) $x_3 \not\sim v_4$ and $x_3 \not\sim v_8$, and by (10) $x_3 \sim v_5$ and $x_3 \sim v_7$. Then $v_1v_2x_3v_5v_4wv_8v_7v_6x_2v_3x_1v_1$ is a Hamilton cycle in G. Therefore $x_3 \not\sim v_2$, and by (10) and (2) $N(x_3) = \{v_4, v_6, v_8\}$. But then $v_1v_2v_3x_1v_5v_4wv_7v_8x_3v_6x_2v_1$ is a Hamilton cycle in G.

$\{v_1, v_3, v_5\} \subset N(x_1)$, $\{v_1, v_3, v_7\} \subset N(x_2)$. Call this Subgraph 1 for later reference.

Note that if $x_3 \sim v_2$, then $x_3 \not\sim v_4$ and $x_3 \not\sim v_8$ by (8) and if $x_3 \sim v_6$, then $v_1v_2x_3v_6v_5v_4wv_8v_7x_2v_3x_1v_1$ is a Hamilton cycle in G. Therefore, if $x_3 \sim v_2$, then $N(x_3) = \{v_2, v_5, v_7\}$. Note by earlier cases and by (2), x_1 and x_2 can have no additional neighbours. Since x_1 and x_3 share only v_5 as a common neighbour, the requirement that $\langle N(v_5)\rangle$ be hamiltonian implies that $d(v_5) \geq 9$.

If $x_3 \not\sim v_2$, then by (10) and (2) $N(x_3) = \{v_4, v_6, v_8\}$, and then $v_1v_2v_3x_1v_5v_4wv_6x_3$ $v_8v_7x_2v_1$ is a Hamilton cycle in G.

Note that by (10), if $x_2 \sim v_1$, then $x_2 \not\sim v_8$, so the next case to consider is

$\{v_1, v_3, v_5\} \subset N(x_1)$, $\{v_1, v_4\} \subset N(x_2)$. Note that $v_5, v_6, v_8 \notin N(x_2)$ by (10) and (8), so that $N(x_2) = \{v_1, v_4, v_7\}$. If $x_3 \sim v_2$, then by (8) $x_3 \not\sim v_4$ and $x_3 \not\sim v_8$, so that $N(x_3) = \{v_2, v_5, v_7\}$, and then $v_1v_2x_3v_5v_6wv_8v_7x_2v_4v_3x_1v_1$ is a Hamilton cycle in G so it follows that $x_3 \not\sim v_2$. If $x_3 \sim v_3$ then x_3 must have two neighbours in $\{v_5, v_6, v_7, v_8\}$, but $x_3 \sim v_6$ results in $v_1v_2v_3x_3v_6wv_8v_7x_2v_4v_5x_1v_1$ and $x_3 \sim v_8$ results in $v_1v_2v_3x_3v_8v_7x_2v_4wv_6v_5x_1v_1$ as Hamiton cycles in G. Therefore $N(x_3) = \{v_3, v_5, v_7\}$. This subgraph (excluding x_2) is isomorphic to the graph labeled Subgraph 1.

Note that by (10), if $x_2 \sim v_1$, then $x_2 \not\sim v_8$, so the next case to consider is

$\{v_1, v_3, v_5\} \subset N(x_1)$, $\{v_1, v_5\} \subset N(x_2)$. In this case $x_2 \not\sim v_6$ and $x_2 \not\sim v_8$ by (10), therefore $N(x_2) = \{v_1, v_5, v_7\}$. Now, if $x_3 \sim v_2$, then $x_3 \not\sim v_1$ by (2), $x_3 \not\sim v_3$ by (10), $x_3 \not\sim v_4$ by (8) and $x_3 \not\sim v_5$ by (2), so then $x_3 \sim v_6$ and $x_3 \sim v_8$, but this is counter to (8). Therefore, $x_3 \not\sim v_2$. The same argument shows that $x_3 \not\sim v_3$. Therefore it must be that $N(x_3) = \{v_4, v_6, v_8\}$, but that is counter to (8).

<u>$\{v_1, v_3, v_5\} \subset N(x_1)$, $\{v_1, v_6\} \subset N(x_2)$</u>. By (10) x_2 can't be adjacent to v_7 or v_8, so this case is not possible.

We can now increment x_2's first neighbour:

<u>$\{v_1, v_3, v_5\} \subset N(x_1)$, $x_2 \sim v_2$</u>. Now $v_3, v_4, v_8 \notin N(x_2)$ by (10) and (8), so that $N(x_2) = \{v_2, v_5, v_7\}$. The same argument shows that if $x_3 \sim v_2$, then $N(x_3) = \{v_2, v_5, v_7\}$. But this is counter to (2). Therefore $x_3 \not\sim v_2$. If $x_3 \sim v_3$ or $x_3 \sim v_4$, then by (2) and (10) $\{v_6, v_8\} \subset N(x_3)$ and $v_1v_2v_3v_4wv_6x_3v_8v_7x_2v_5x_1v_1$ is a Hamilton cycle in G. Therefore this case is not possible.

<u>$\{v_1, v_3, v_5\} \subset N(x_1)$, $x_2 \sim v_3$</u>. By previous cases and (2) and (10), $N(x_3) = \{v_4, v_6, v_8\}$, but this is counter to (8).

<u>$\{v_1, v_3, v_5\} \subset N(x_1)$, $x_2 \sim v_4$</u>. By (10) $N(x_2) = \{v_4, v_6, v_8\}$ which is counter to (8).

We must therefore increment the neighbours of x_1. Based on the cases studied up to this point, we can add another claim.

<u>Claim 11:</u> No vertex in X can be adjacent to v_i, v_{i+2} and v_{i+4}.

By (10) the next iteration is

<u>$\{v_1, v_3, v_6\} \subset N(x_1)$, $\{v_1, v_3\} \subset N(x_2)$</u>. By (10) and (11), $v_4, v_5, v_7, v_8 \notin N(x_2)$. Therefore $x_2 \sim v_6$, and by (10) and (2), x_3 is adjacent to one of $\{v_4, v_5\}$ and to one of $\{v_7, v_8\}$, implying that $x_3 \sim v_2$ by (2), so that $x_3 \not\sim v_4$ by (8). Then $x_3 \sim v_5$, and by (8) $x_3 \not\sim v_7$, so that $x_3 \sim v_8$, but this is also against (8).

<u>$\{v_1, v_3, v_6\} \subset N(x_1)$, $\{v_1, v_4\} \subset N(x_2)$</u>. If $x_3 \sim v_2$, then $v_3, v_4, v_8 \notin N(x_3)$ by (8) and (10), so that $N(x_3) = \{v_2, v_5, v_7\}$. Then by (8), $x_2 \not\sim v_6$, so that $N(x_2) = \{v_1, v_4, v_7\}$, but then $v_1x_1v_3v_2x_3v_7v_8wv_6\ v_5v_4x_2v_1$ is a Hamilton cycle in G. Therefore $x_3 \not\sim v_2$. It follows that $x_3 \sim v_3$, else $N(x_3) = \{v_4, v_6, v_8\}$, which is counter to (11). Then if $x_3 \sim v_5$, $v_1v_2wv_8v_7v_6x_1v_3x_3v_5v_4x_2v_1$ is a Hamilton cycle in G. So $\{v_6, v_8\} \subset N(x_3)$. But then $v_1v_2v_3x_1v_6x_3v_8v_7wv_5v_4x_2v_1$ is a Hamilton cycle in G.

<u>$\{v_1, v_3, v_6\} \subset N(x_1)$, $\{v_1, v_5\} \subset N(x_2)$</u>. From (10) we know that the remaining neighbour of x_2 is v_7. If $x_3 \sim v_2$, then by (10), $x_3 \not\sim v_3$ and by (8) $x_3 \not\sim v_4$ and $x_3 \not\sim v_8$, so by (10) we get $N(x_3) = \{v_2, v_5, v_7\}$, but then $v_1v_2x_3v_5v_4v_3x_1v_6$ $wv_8v_7x_2v_1$ is a Hamilton cycle in G, so $x_3 \not\sim v_2$. Now if $x_3 \sim v_3$ and $x_3 \sim v_8$, then $v_1v_2v_3x_3v_8v_7x_2v_5v_4wv_6x_1v_1$ is a Hamilton cycle in G. If $x_3 \not\sim v_8$, then by (10) $N(x_3) = \{v_3, v_5, v_7\}$, but this is counter to (11), so $x_3 \not\sim v_3$, and by (10) and (11),

v_4 can't be the first neighbour of x_3, so we must increment the neighbours of x_2.

Note that if the second neighbour of x_2 is v_6, then the third neighbour must be v_8, which is counter to (10). By (10) the next iteration is

$\{v_1, v_3, v_6\} \subset N(x_1),\ x_2 \sim v_2$. By (10), $x_2 \not\sim v_3$ and by (8), $x_2 \not\sim v_4$ and $x_2 \not\sim v_8$, so it follows that $\{v_2, v_5, v_7\} = N(x_2)$. If $x_3 \sim v_2$ then by (10) and (8), $N(x_3) = \{v_2, v_5, v_7\}$ and $v_1v_8v_7x_3v_2x_2v_5v_4\ v_3wv_6x_1v_1$ is a Hamilton cycle in G, and so $x_3 \not\sim v_2$. If $x_3 \sim v_3$ and $x_3 \sim v_5$, then $v_1v_2x_2v_5x_3v_3v_4wv_8v_7v_6x_1v_1$ is a Hamilton cycle in G, and if $x_3 \sim v_3$ and $x_3 \sim v_6$, then $v_1v_2x_2v_5v_4wv_8v_7v_6x_3v_3x_1v_1$ is a Hamilton cycle in G, so by (8) and (10) $x_3 \not\sim v_3$. Then by (10) and (11) $x_3 \not\sim v_4$, and therefore $x_2 \not\sim v_5$, so we have a contradiction.

$\{v_1, v_3, v_6\} \subset N(x_1),\ x_2 \sim v_3$. In this case by (2) and (10) we must have $N(x_3) = \{v_4, v_6, v_8\}$, which is counter to (11).

By (10) and (11) it is not possible that the first neighbour of x_2 is v_4, so we must increment the neighbours of x_1, but by symmetry all the possibilities have already been exhausted. We have now completed the proof that if $|X| = 3$ and the vertices in X are independent, and $n(G) = 12$, then $\Delta(G) \neq 8$.

Note that the proof up to this point only depends on the fact that for an isolated vertex x in X, there are at least three edges between $N(w)$ and x, and that a Hamilton cycle can go through x via any two of these edges. If the vertex x is replaced by a pair with an edge between them, x_1x_2, the same holds. To see this, note that local hamiltonicity requires that the two vertices x_1 and x_2 must have at least two neighbours in $N(w)$, say v_i and v_j, in common, and the requirement of G being 3-connected implies that at least one of them, say x_1, must have third neighbour, say v_k. Now a Hamilton cycle can go through x_1x_2 via any two of these three edges: $v_kx_1x_2v_i$, $v_kx_1x_2v_j$, $v_ix_1x_2v_j$. This means that if a section of the proof holds for X consisting of m isolated vertices, the same proof will hold if $comp(X) = m$, where one or more of the components of X consists of a K_2, and the other components of X are isolated vertices. We will refer to this as Claim 12.

By Claim 12 it follows the above proof for $\Delta = 8$ and $|X| = 3$ also holds for $\Delta = 7$ and $comp(X) = 3$, except that there are fewer cases to consider.

We now proceed to consider the case where $\Delta(G) = 8$ and $comp(X) = 2$, that is, the edge x_1x_2 is in $E(G)$.

We have some additional claims for this part of the proof:

Claim 13: From (12) it follows that x_1 and x_2 share at least two neighbours in $N(w)$ and $|N(w) \cap N(x_1) \cap N(x_2)| \geq 3$.

Claim 14: If S is a Hamilton path of a component of X, then in G the path $v_i S v_{i+1}$ is not in any component in X, where the indices are taken modulo 8.

Proof of Claim 14: Since $N[w]$ is traceable between any two vertices in $N(w)$, and $\delta(G) \geq 3$, the result follows.

Claim 15: For no vertex v_i in $N(w)$ do we have $\{x_1, x_2, x_3\} \subset N(v_i)$.

Proof of Claim 15: If $N(v_i) \subset \{x_1, x_2, x_3\}$, then by Claim 14 v_{i-1} and v_{i+1} are not adjacent to any vertices in X. Since *comp*$(X) = 2$, and $\langle N(v_i) \rangle$ is hamiltonian, it follows that $|N(v_i) \cap N(w)| \geq 5$ which implies that $d(v_i) \geq 9$.

Now, if $\{v_1, v_3\} \subset N(x_1) \cap N(x_2)$, then if $x_3 \not\sim v_2$, then $N(x_3) = \{v_4, v_6, v_8\}$, but then $v_1 x_1 x_2 v_3 v_2 w v_7 v_6 v_5 v_4 x_3 v_8 v_1$ is a Hamilton path in G. So $v_2 \sim x_3$. But then if v_i is a second neighbour of x_3, $i \in \{4, 5, 6, 7, 8\}$, $v_1 x_1 x_2 v_3 v_2 x_3 v_i v_{i-1} \ldots v_4$ $w v_{i+1} v_{i+2} \ldots v_8 v_1$ is a Hamilton path in G. So the neighbours that x_1 and x_2 have in common are not at a distance of two in C.

If $\{v_1, v_4\} \subset N(x_1) \cap N(x_2)$, then by (14) and (15) $x_3 \sim v_2$ or $x_3 \sim v_3$. Without loss of generality let $x_3 \sim v_2$. Then x_3 must have a second neighbour v_i, $i \in \{5, 6, 7, 8\}$. But then $v_1 x_1 x_2 v_4 v_3 v_2 x_3 v_i v_{i-1} \ldots v_5 w\ v_{i+1} v_{i+2} \ldots v_8 v_1$ is a Hamilton path in G. Therefore the neighbours that x_1 and x_2 have in common are at a distance of four in $N(w)$.

If v_1 and v_5 are the two neighbours that x_1 and x_2 have in common and x_3 is adjacent to any of v_2, v_4, v_6 and v_8 then a Hamilton cycle in G can be found in the same way as in the previous case. But then x_3 can have only two neighbours. Therefore it is not possible that $\Delta(G) = 8$ if *comp*$(X) = 2$, and G is obviously hamiltonian if *comp*$(X) = 1$.

This leaves the case where $\Delta(G) = 7$ and $|X| = 4$. First we consider the subcase where *comp*$(X) = 4$, and we make some fresh claims.

Claim 16: For $i \in \{1, 2, \ldots, 7\}$, $|N(v_i) \cap V(X)| \leq 2$ and if $\{x_j, x_k\} \subset N(v_i)$, $j \neq k$, then $v_{i-1} \sim x_j$ and/or $v_{i+1} \sim x_j$ (the latter requirement will be referred to as (16a)).

Proof of Claim 16: This follows directly from the fact that the vertices in X are independent, that $\langle N(v_i)\rangle$ is hamiltonian, and that $\Delta(G) = 7$.

Claim 17: If $\{v_i, v_{i+1}\} \subset N(x_q)$ and $\{v_j, v_{j+1}\} \subset N(x_p)$, $i \neq j$, then if $\{v_k, v_{k+1}\} \subset N(x_r)$, $q \neq p \neq r$, then $k \in \{i, j\}$.

Proof of Claim 17: The result follows from (1) and the facts that $\delta(G) \geq 3$ and $N[w]$ is traceable between any two vertices in $N(w)$.

Claim 18: If $\{v_i, v_{i+1}\} \subset N(x_q)$ and $\{v_j, v_{j+1}\} \subset N(x_p)$ where $i \neq j$, and $x_r \sim v_k$, and $x_t \sim v_{k+1}$, $p \neq q \neq r \neq t$, then $k \in \{i, j\}$.

Proof of Claim 18: Again the result follows from (1) and the facts that $\delta(G) \geq 3$ and $N[w]$ is traceable between any two vertices in $N(w)$.

Claim 19: There is no subgraph of G in which $\{v_i, v_{i+1}\} \subset N(x_p)$, $\{v_i\} \subset N(x_q)$, $\{v_j, v_{j+1}\} \subset N(x_r)$, and $\{v_j\} \subset N(x_t)$, $i \neq j$, $p \neq q \neq r \neq t$.

Proof of Claim 19: If $\{v_1, v_2\} \subset N(x_1)$, $\{v_1\} \subset N(x_2)$, $\{v_3, v_4\} \subset N(x_3)$ and $\{v_3\} \subset N(x_4)$, then $x_2 \not\sim v_7$ and $x_4 \not\sim v_2$ by (17) and $x_2 \not\sim v_2$ and $x_4 \not\sim v_7$ by (18) and $x_2 \not\sim v_3$ and $x_4 \not\sim v_1$ by (17). Therefore x_2 and x_4 must each have at least two neighbours in $\{v_4, v_5, v_6\}$, so by (17), $N(x_2) = \{v_1, v_4, v_6\}$, which means that by (16) $N(x_4) = \{v_3, v_5, v_6\}$, which is counter to (17). This scenario is therefore not possible.

Let $\{v_1, v_2\} \subset N(x_1)$, $\{v_1\} \subset N(x_2)$, $\{v_4, v_5\} \subset N(x_3)$, $\{v_4\} \subset N(x_4)$. Then $x_2 \not\sim v_7$ and $x_4 \not\sim v_3$ by (17) and $x_2 \not\sim v_3$ and $x_4 \not\sim v_7$ by (18). Therefore by (16) and (17) $x_2 \sim v_2$, and by (16) $\{v_5, v_6\} \subset N(x_4)$, which is counter to (17).

By symmetry, this exhausts the possibilities and the proof of the claim is complete.

Claim 20: There is no subgraph of G in which $\{x_p, x_q\} \subset N(v_i)$, $\{x_r, x_t\} \subset N(v_j)$, $i \neq j$, $p \neq q \neq r \neq t$.

Proof of Claim 20: Note that if $\{x_1, x_2\} \subset N(v_1)$, then by (16a) without loss of generality let $v_2 \sim x_1$ and then $\{x_3, x_4\} \not\subset N(v_2)$ by (16a). If $\{x_1, x_2\} \subset N(v_1)$, $v_2 \sim x_1$ and $\{x_3, x_4\} \subset N(v_3)$ then by (16) and (19) we can say without loss of generality that $v_2 \sim x_3$. Then by (16) $v_2 \not\sim x_2$, $v_3 \not\sim x_2$, by (17) $v_7 \not\sim x_2$ and by (18) $v_4 \not\sim x_2$. Then $\{v_5, v_6\} \subset N(x_2)$, which is counter to (17).

Now if $\{x_1, x_2\} \subset N(v_1)$, $v_2 \sim x_1$ and $\{x_3, x_4\} \subset N(v_4)$ then by (16a) and (19) we can say without loss of generality that $v_3 \sim x_3$. Then $v_1 \not\sim x_4$ by (1), $v_5 \not\sim x_4$ by (17) and $v_7 \not\sim x_4$ by (18), so that by (17) $x_4 \sim v_6$ and x_4 is adjacent to one of v_2

and v_3. Then by (18) $v_5 \not\sim x_2$ and by (17) $v_7 \not\sim x_2$ and by (16) $v_4 \not\sim x_2$, so that by (17) $x_2 \sim v_6$ and x_2 is adjacent to one of v_2 and v_3. But then one of v_2 and v_3 has three neighbours in $V(X)$, counter to (16).

Now if $\{x_1, x_2\} \subset N(v_1)$, $v_2 \sim x_1$ and $\{x_3, x_4\} \subset N(v_5)$ then by (16a) and (19) we can say without loss of generality that $v_4 \sim x_3$. Then by (16) $v_1 \not\sim x_4$, by (2) $v_6 \not\sim x_4$, and by (18) $v_7 \not\sim x_4$, so that by (17) $\{v_2, v_4\} \subset N(x_4)$, which implies by (16) and (17) that $x_2 \sim v_3$, which is counter to (18).

Now if $\{x_1, x_2\} \subset N(v_1)$, $v_2 \sim x_1$ and $\{x_3, x_4\} \subset N(v_6)$ then by (16a) and (19) we can say without loss of generality that $v_5 \sim x_3$. By (17) $x_2 \not\sim v_7$ and $x_4 \not\sim v_7$. Thus by (16) x_2 and x_4 must each have two neighbours in $\{v_2, v_3, v_4, v_5\}$, and by (17) and (18) these neighbours may not be successive in C. This implies that x_2 and x_4 must have the same two neighbours in $\{v_2, v_3, v_4, v_5\}$. But this is not possible by (16) and (18).

Now if $\{x_1, x_2\} \subset N(v_1)$, $v_2 \sim x_1$ and $\{x_3, x_4\} \subset N(v_7)$ then by (16a) and (19) we can say without loss of generality that $v_6 \sim x_3$. This is counter to (18).

By symmetry, this exhausts the possibilities and the proof of the claim is complete.

We will now attempt to allocate three edges to $N(w)$ from each vertex in X. We will rename the vertices in $N(w)$ to make it clear that the sequence of vertices in a possible cycle is not relevant here: $N(w) = \{a, b, c, d, e, f, g\}$. Without loss of generality (since there have to be twelve edges incident to the seven vertices in $N(w)$), suppose ax_1 and ax_2 are edges in G. Then by (20) x_3 and x_4 can't share any neighbours.

Then if we assume that x_1 and x_2 do not share a second neighbour, we can assume the following: $N(x_1) = \{a, b, c\}$ and $N(x_2) = \{a, d, e\}$. Then if $b \sim x_3$, by (20) x_2 and x_4 can't share any neighbours, so $N(x_4) = \{c, f, g\}$, which means that x_1 and x_4 share a neighbour, so x_2 and x_3 do not share neighbours. Therefore there is no possible third neighbour for x_3. We can then conclude by symmetry that x_1 and x_2 do not share any neighbours with x_3 and x_4. But then the only possible neighbours for x_3 and x_4 are f and g. Therefore x_1 and x_2 must share at least two neighbours.

Now assume x_1 and x_2 are both adjacent to a and b. Then x_3 and x_4 still can't have any neighbours in common, but there are only five vertices (c, d, e, f, g) available for them to have as neighbours. So x_1 and x_2 can't share more than one neighbour. Therefore we conclude that if $\Delta(G) = 7$ and $comp(X) = 4$, then G cannot be nonhamiltonian and LH.

All that remains is to address the cases where $\Delta(G) = 7$ and $comp(X) < 4$.

By (12) the only scenarios that we still have to address are the ones where $comp(X) \leq 2$ and none of the components of X is a connected pair.

Since G is obviously hamiltonian if X has only one component that can be traced between two vertices that have distinct neighbours in $N(w)$, there remain three cases to consider: X contains either the path $x_2x_3x_4$, or K_3, or the claw $K_{1,3}$. In all three cases there is a two-path cover for X of which one of the paths is a singleton vertex, call it x_1, and the other path can be labeled $x_2x_3x_4$. We start by making two new claims. The proofs of the claims follow readily from the facts that $N[w]$ is traceable between any two vertices in $N(w)$, $\delta(G) \geq 3$, and G is 3-connected, and are not presented here.

Claim 21: x_1 can't have successive neighbours in C and if $x_2 \sim v_i$, then $x_4 \not\sim v_{i-1}$ and $x_4 \not\sim v_{i+1}$.

Claim 22: If $\{v_i, v_{i+2}\} \subset N(x_1)$, then $x_2 \not\sim v_{i+1}$ and $x_4 \not\sim v_{i+1}$.

Case 1: $X = \{x_1, x_2x_3x_4\}$. By (21) we can say without loss of generality that $N(x_1) = \{v_1, v_3, v_5\}$. By (22) it follows that $v_2, v_4 \notin N(x_2)$ and $v_2, v_4 \notin N(x_4)$. If $x_2 \sim v_7$, then $v_1, v_6 \notin N(x_4)$ by (21) and if $x_4 \sim v_3$, then $v_1v_2v_3x_4x_3x_2v_7v_6wv_4v_5x_1v_1$ is a Hamilton cycle in G, and if $x_4 \sim v_5$, then $v_1v_2v_3v_4wv_6v_7x_2x_3x_4v_5x_1v_1$ is a Hamilton cycle in G. Therefore, by symmetry, $v_6, v_7 \notin N(x_2)$ and $v_6, v_7 \notin N(x_4)$, so that $N(w) \cap (N(x_2) \cup N(x_4)) \subset \{v_1, v_3, v_5\}$. But each of x_2 and x_4 has at least two neighbours in $N(w)$, and if $v_i \in N(x_1) \cap N(x_2) \cap N(x_4)$, then by (21) $d(v_i) \geq 9$. Therefore this case is not possible.

Case 2: $X = \{x_1, K_3\}$. From the argument in Case 1 it follows that $N(x_1) = \{v_1, v_3, v_5\}$, and that without loss of generality we can claim that $x_2 \sim v_1$, $x_3 \sim v_3$ and $x_4 \sim v_5$. But now if we consider $\langle N(v_1) \rangle$ it is clear that since $x_1 \not\sim x_2$, local hamiltonicity requires $d(v_1) \geq 8$.

Case 3: $X = K_{1,3}$. Let x_3 be the vertex of degree 3 in X. By (21) and symmetry in X it follows that no vertex in $\{x_1, x_2, x_4\}$ can be adjacent to successive vertices in C, and if $x_j \sim v_i$, then $x_k \not\sim v_{i+1}$, $j, k \in \{1, 2, 4\}$, $j \neq k$. So without loss of generality, we can say that the vertices in $\{v_1, v_3, v_5\}$ are each adjacent to two elements of $\{x_1, x_2, x_4\}$. Since $\{x_1, x_2, x_4\}$ is an independent set, the hamiltonicity of say, $\langle N(v_1)\rangle$, requires that $d(v_1) \geq 8$.

This completes the proof. $\square$

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